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CONTENTS.

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THE

UNIVERSAL CALCULATOR'S

POCKET GUIDE;

A COMPANION

TO EVERY SET OF

MATHEMATICAL TABLES,

SHOWING THEIR CONSTRUCTION AND APPLICATION TO ARITHMETIC, MENSURATION, TRIGONOMETRY, SURVEYING, NAVIGATION, ASTRONOMY, &c., &c.

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PREFACE.

THIS little work is intended as a suitable "Companion to every set of Mathematical Tables," but especially to those contained in the "*Practical Mathematician's Pocket Guide.*" The first edition of that work was accompanied by an introduction, explaining the nature and use of logarithms, and the mode of their application to practice. The whole of that introduction, excepting the explanation of the tables, (which is prefixed to the above-mentioned work) is given in the present work, and a great deal of new and useful matter is added, in order to render it a complete directory for the use of practical men.

The introduction contains the explanation of the principles of the construction and use of logarithms, and several methods of calculating them by common arithmetic, some of which are indeed founded on purely arithmetical principles.

The first section contains Mathematical investigations of the rules given in the introduction, and of the most useful Algebraic formulæ relating to logarithms and to trigonometrical functions connected with them in practice. The former investigations are conducted on principles much more simple than those commonly to be found

in works written expressly on the subject; and the latter are established on the principles of the Differential Calculus, by the application of Taylor's Theorem.

The first part of the second section contains the application of logarithms to the purposes of common arithmetic. Besides the usual rules, the subject of Compound Interest and Annuities is concisely but distinctly treated.

The second part contains a treatise on Mensuration by logarithms, including Lines, Surfaces, and Solids. Here also, besides the usual rules, some problems of rather a novel and useful description will be found interspersed.

The third part contains a plain and concise set of logarithmic rules for the resolution of the most general and useful problems in Plane and Spherical trigonometry.

The first part of the third section contains an investigation of the rules and formulæ of the most common and general utility in Plane trigonometry. Though this investigation be necessarily conducted in a condensed form, yet it is hoped that it will be found sufficiently clear and interesting to the attentive student; and though much that is new cannot be expected under this head, yet the mode of treating the subject, will be found considerably different from that of the common treatises.

The second part treats of the Mensuration of Heights and Distances, in a manner calculated to interest the learner more than usual, the rules of trigonometry being immediately applied to some of the most useful and practical questions in Astronomy, Surveying, &c.

The third part contains an investigation of the Rules and Formulæ of Spherical Trigonometry, conducted on the same condensed but novel plan pursued in the former parts of this section; a very prominent place being given to Napier's admirable Rules for the Circular Parts, with his Spherical Analogies, &c.

The fourth part contains the application of the principles of Spherics to the solution of a great variety of useful and interesting problems in Astronomy and Navigation, including the method of finding the longitude at sea by Chronometers, and by Lunar Observations. The rule for the latter mode is simple, and possesses several advantages over those in common treatises on navigation, which require, in working them, the consultation of too many tables of different kinds; in the application of this rule, one table is sufficient.

The fifth part contains a curious rule in Surveying, by which the areas or superficial contents of fields can be found by the tables of Difference of Latitude and Departure given in books of navigation; a small but useful table of this kind is given in this work.

The fourth and last section contains a number of tables of the greatest use to surveyors, mariners, astronomers, engineers, &c.; these tables were originally intended to be appended to those contained in the "*Practical Mathematician's Pocket Guide*," but they were retained for this work, to admit of the former being sold at the very cheap rate at which the Publisher now offers it to the public. The two works combined, will, it is hoped, be found of great utility to all classes of

the community, but especially to those whose pursuits lead them to the practical application of the Mathematical Sciences in all their departments.

In fine, to anticipate the literary critics, the Author thinks it proper to mention that in his derivation of the word logarithms, he has used a freedom which he considers perfectly lawful, in order to convey a correct notion of the term, according to its more modern acceptation. This remark will be understood by those who are conversant with the history of the invention of logarithms. The Author's only object in making it, is to avoid the unnecessary imputation of gross ignorance, either of that history, or of the rich and elegant language from which the term was derived.

BLYTHSWOOD HILL MATHEMATICAL ACADEMY,
GLASGOW, 1st *December*, 1838.

CONTENTS.

SECTION III.—INVESTIGATION OF THE RULES.

II. MENSURATION of HEIGHTS and DISTANCES (*continued*).

III. SPHERICAL TRIGONOMETRY.

IV. ASTRONOMICAL and NAUTICAL PROBLEMS.

PAGE

SECTION IV.—ASTRONOMICAL AND NAUTICAL TABLES.

INTRODUCTION.

1. The term *Logarithms* is derived from two Greek words, (*logoi* and *arithmon*,) and literally signifies *indices of numbers*. Hence, logarithms are defined, a system of *artificial* numbers which *indicate*, or represent the *natural* numbers.

2. By the use of logarithms instead of the natural numbers, the tedious and difficult calculations of arithmetic may be superseded; the irksome and laborious operations of multiplication and division may be effected by simple additions and subtractions; and the operose and intricate processes of involution and evolution, or the raising of powers and the extraction of roots, may be reduced to simple multiplications and divisions.

NATURE OF POWERS AND ROOTS.

3. If a number be multiplied by itself, the product is called the *square* or *second power;* and the number itself, the *root* or *first power*. If the second power be multiplied by the root, the product is called the *cube* or *third power*. If the third power be multiplied by the root, the product is called the *fourth power;* and so on. Thus, the multiplication of a number by itself, and by each successive product, produces the powers of that number. This process is called *involution*, or raising of powers.

4. The numbers which *indicate* how often the root enters as a factor into each of the powers, are called the *indices* of those powers. Thus, the index of the first power is 1; of the second power, 2; of the third power, 3; of the fourth power, 4; and so on.

5. If a number be considered as a power of another, the latter is called *such a root* of the former as is denoted by the *name* of that power. Thus, if a number be considered as the *square* of another, the latter is called the *square root* of the former. If a number be considered as the *cube* of another, the latter is called the *cube root* of the former. If a number be considered as the *fourth power* of another, the latter is called the *fourth root* of the former; and so on. The process by which the roots are found, is called *evolution*

or the extraction of roots, and is explained in the common treatises on arithmetic.

6. The *reciprocal* of a number is a fraction whose numerator is unity, and whose denominator is the number itself. Thus, the reciprocal of 2 is $\frac{1}{2}$; of 3 is $\frac{1}{3}$; of 4 is $\frac{1}{4}$; and so on.

7. The *reciprocals* of the numbers which *indicate* how often the root of a number enters as a factor into that number, are called the *indices* of its roots. Thus, the index of the square root is $\frac{1}{2}$; of the cube root, $\frac{1}{3}$; of the fourth root, $\frac{1}{4}$; and so on. The indices of the roots of a number being fractions, the *roots* themselves are called *fractional powers* of that number.

8. If the number 2 be assumed as the root or first power, then the successive powers of 2, are calculated by simple multiplication in the following manner:

$2=$	2	First Power
$2\times 2=$	4	Second Power
$4\times 2=$	8	Third Power
$8\times 2=$	16	Fourth Power
$16\times 2=$	32	Fifth Power
$32\times 2=$	64	Sixth Power
$64\times 2=$	128	Seventh Power
$128\times 2=$	256	Eighth Power
$256\times 2=$	512	Ninth Power
$512\times 2=$	1024	Tenth Power
&c.	&c.	&c.

These powers of the number 2 are denoted by placing their indices on the right of the root, as in the following

Table of Powers.

$2^1=$	2	Root or First Power
$2^2=$	4	Square or Second Power
$2^3=$	8	Cube or Third Power
$2^4=$	16	Fourth Power
$2^5=$	32	Fifth Power
$2^6=$	64	Sixth Power
$2^7=$	128	Seventh Power
$2^8=$	256	Eighth Power
$2^9=$	512	Ninth Power
$2^{10}=$	1024	Tenth Power
&c.	&c.	&c.

9. The roots of these powers are, in like manner, denoted by placing their indices on the right of the powers, each being considered a separate number, as in the following

Table of Roots.

$4^{\frac{1}{2}} = 2$, Square root of 4

$8^{\frac{1}{3}} = 2$, Cube root of 8

$16^{\frac{1}{4}} = 2$, Fourth Root of 16

$32^{\frac{1}{5}} = 2$, Fifth root of 32

$64^{\frac{1}{6}} = 2$, Sixth root of 64

$128^{\frac{1}{7}} = 2$, Seventh root of 128

$256^{\frac{1}{8}} = 2$, Eighth root of 256

$512^{\frac{1}{9}} = 2$, Ninth root of 512

$1024^{\frac{1}{10}} = 2$, Tenth root of 1024

&c. &c. &c.

10. The *multiplication* of the powers or roots of the same number, is indicated by the *addition* of the indices of the factors. Thus, if 16, the fourth power of 2, be multiplied by 8, its third power, the product 128, is its seventh power. This process is thus indicated: if 4 and 3, the indices of the factors, be added together, the sum 7 is the index of the product. Again, if 4, the cube root of 64, be multiplied by 2, its sixth root, the product 8, is its square root. This process is thus indicated: if $\frac{1}{3}$ and $\frac{1}{6}$, the indices of the factors be added together, the sum $\frac{1}{2}$, is the index of the product.

11. The *division* of powers or roots of the same number is indicated by the *subtraction* of the index of the divisor from that of the dividend. Thus, if 512, the ninth power of 2, be divided by 128, its seventh power, the quotient 4, is its square. This process is thus indicated: if 7, the index of the divisor, be subtracted from 9, the index of the dividend, the remainder 2, is the index of the quotient. Again if 8, the square root of 64, be divided by 4, its cube root, the quotient 2, is its sixth root. This process is thus indicated: if $\frac{1}{3}$, the index of the divisor, be subtracted from $\frac{1}{2}$, the index of the dividend, the remainder $\frac{1}{6}$, is the index of the quotient.

12. The *involution* of powers or roots of the same number is indicated by the *multiplication* of the indices of the given powers or roots by those of the required powers. Thus, if 16, the fourth power of 2 be squared, the square 256 is the eighth

power of 2. This process is thus indicated; if 4, the index of the fourth power of 2, be multiplied by 2, the index of the square, the product 8, is the index of the required power of 2. Again, if 2 the sixth root of 64, be cubed, the cube 8, is the square root of 64. This process is thus indicated: if $\frac{1}{6}$, the index of the sixth root, be multiplied by 3, the index of the cube, the product $\frac{1}{2}$ is the index of the required power of 64.

13. The *evolution* of powers or roots of the same number is indicated by the *multiplication* of the indices of the given powers or roots, by the indices of the required roots; or, by the *division* of those indices by the *denominators* of the indices of the required roots. Thus, if the cube root of 512, the ninth power of 2, be extracted, the cube root 8, is the third power of 2. This process is thus indicated: if 9, the index of the ninth power, be divided by 3, the denominator of the index of the cube root, the quotient 3, is the index of the required power of 2. Again, if the square root of 4, which is the same root of 16, be extracted, the square root 2, is the fourth root of 16. This process is thus indicated: if $\frac{1}{2}$, the index of the given root be multiplied by $\frac{1}{2}$, the index of the required root of 4, the product $\frac{1}{4}$ is the required root of 16.

14. If any power or root of a number be divided by itself, the quotient is unity. On the principle, that the division of powers and roots of the same number is indicated by subtracting the index of the divisor from that of the dividend, the index of the quotient unity, in this case is 0. Hence, that power of any number, whose index is 0, is unity. Consequently, in the expression $2^0 = 1$, unity may be denominated the *zero* power of 2.

NATURE AND USE OF LOGARITHMS.

15. The powers, of any assumed invariable number or root, constitute a series of numbers which are denominated the *natural numbers*. The *indices* of those powers are called *artificial numbers* or *logarithms*; and taken collectively, a *system of logarithms*. The assumed constant number or *root*, is called the *base* of the system.

16. The *logarithm* of a given number is precisely defined, the *index* of that power of the *base* of the system, which is equal to the given number.

17. Any number may be assumed as the base of a system of logarithms. If the number 2, be assumed as the *base*, then the powers of 2 will be the *natural numbers*, and the *indices* of those powers will be the *logarithms* of the natural numbers. If these definitions be applied to the numbers and their indices in the table of powers, art. 8, the expression in art. 14, being also taken into consideration, the construction of the following table will be rendered manifest:

Table of Logarithms.

Natural Numbers.	Logarithms.
1	0
2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10
&c.	&c.

By means of this table, logarithmic calculations may be exemplified on a small scale, in the following manner.

18. First, To multiply two or more numbers together. If the logarithms of the factors be added together, the sum is the logarithm of the product, (see art. 10). Thus, to multiply 128 by 8, add 7 and 3 together, the logarithms of the factors, the sum 10 is the logarithm of the product 1024. Again, to multiply 4, 8 and 16 continuously together, add 2, 3 and 4, together, the logarithms of the factors, the sum 9 is the logarithm of the product 512.

19. Secondly, To divide one number by another. If the logarithm of the divisor be subtracted from the logarithm of the dividend, the remainder is the logarithm of the quotient (see art. 11). Thus, to divide 256 by 64, subtract 6, the logarithm of the divisor from 8, the logarithm of the dividend, the remainder 2 is the logarithm of the quotient 4.

20. 3°. To find a fourth proportional to three given terms. If the logarithms of the second and third terms be added together, and from the sum, the logarithm of the first term be subtracted, the remainder is the logarithm of the fourth term. Thus, to find a fourth proportional to 8, 32 and 64; if 8 : 32 : : 64 : the fourth term, then add 5 and 6 together, the logarithms of the second and third terms, and from the sum 11, subtract 3, the logarithm of the first term, the remainder 8 is the logarithm of the fourth term 256.

21. Fourthly, To find any power of a number. If the logarithm of the number be multiplied by the index of the required power, the product is the logarithm of that power (see art. 12). Thus, to find the square of 16, multiply 4, the logarithm of the number, by 2 the index of the square, the product 8 is the logarithm of the square 256.

22. Fifthly, To find any root of a number. If the logarithm of the number be multiplied by the index of the required root, or be divided by its denominator, the quotient

is the logarithm of that root (see art. 13). Thus, to find the cube root of 64, divide 6, the logarithm of the number, by 3, the denominator of the index of the cube root the quotient 2 is the logarithm of the cube root 4.

23. The nature and use of logarithms having been thus illustrated and exemplified in the system of which the base is 2, it only remains to give a full explanation of the system in common use.

COMMON SYSTEM.

24. THE number 10 has been assumed as the *base* of the common system of logarithms, because it is the *root* of the decimal scale of notation, and on this account, possesses certain advantages which have led to its universal adoption.

25. The powers of the number 10, being respectively unity with as many ciphers annexed, as are denoted by the indices of the different powers, the construction of the following table is sufficiently manifest:

Table of Powers.

10^0	$= 1$	Zero Power.
10^1	$= 10$	First Power.
10^2	$= 100$	Second Power
10^3	$= 1000$	Third Power.
10^4	$= 10000$	Fourth Power.
10^5	$= 100000$	Fifth Power.
10^6	$= 1000000$	Sixth Power.
10^7	$= 10000000$	Seventh Power.
10^8	$= 100000000$	Eighth Power.
10^9	$= 1000000000$	Ninth Power.
10^{10}	$= 10000000000$	Tenth Power
&c.	&c.	&c.

26. These powers of 10 being the *natural numbers*, and their *indices* the *logarithms* of those numbers, the construction of the following table is rendered evident by the preceding article:

First Skeleton Table of Logarithms.

Natural Numbers.	Logarithms.
1	0
10	1
100	2
1000	3
10000	4
100000	5
1000000	6
10000000	7
100000000	8
1000000000	9
10000000000	10
&c.	&c.

27. If unity, the first natural number, be divided by the successive natural numbers in the preceding table, the quotients will be a series of decimal fractions, viz. .1, .01, .001, &c. The logarithms of these quotients will be found by subtracting the logarithms of the natural numbers from 0, the logarithm of unity (see art. 11). Now, though it be impossible, arithmetically, to subtract the logarithms, 1, 2, 3, &c., from the logarithm 0, yet the operation that should be performed is indicated by placing the sign of subtraction before each of these logarithms; thus—1, —2, —3, &c. Hence, the construction of the following table of decimal fractions, with their logarithms, is evident;

Second Skeleton Table of Logarithms.

Natural Numbers.	Logarithms.
.1	—1
.01	—2
.001	—3
.0001	—4
.00001	—5
.000001	—6
.0000001	—7
.00000001	—8
.000000001	—9
.0000000001	—10
&c.	&c.

28. These logarithms being of an opposite character to the former, are called *negative*, while the former are denominated *positive*. From the preceding article, it is evident that the logarithm of every proper fraction is essentially negative, and that the logarithms of such fractions numerically increase in proportion as the fractions themselves decrease in value, compared with unity. Hence, when the value of a fraction is indefinitely small, its logarithm numerically considered must be indefinitely great; and when the value of a fraction is infinitely small, so as to be reckoned equal to nothing, its logarithm must be infinitely great; in other words, the logarithm of 0, is negative infinity.

29. If the square root of the number 10 be extracted, and then the square root of this root, and of each successive root, the indices of these roots will be the successive powers of $\frac{1}{2}$, the index of the square root (see art. 12.) Thus, by the common rule for extracting the square root, we have,

Square root of 10.00000 = 3.16228, index $\frac{1}{2}$
Square root of 3.16228 = 1.77828, index $\frac{1}{4}$
Square root of 1.77828 = 1.33352, index $\frac{1}{8}$
Square root of 1.33352 = 1.15478, index $\frac{1}{16}$
&c. &c. &c. &c.

On this principle, the following table is constructed:

Table of Even Roots.

$10^{\frac{1}{2}} = 3.16228$ Square root.

$10^{\frac{1}{4}} = 1.77828$ Fourth root.

$10^{\frac{1}{8}} = 1.33352$ Eighth root.

$10^{\frac{1}{16}} = 1.15478$ Sixteenth root.

$10^{\frac{1}{32}} = 1.07461$ Thirty-second root.

$10^{\frac{1}{64}} = 1.03663$ Sixty-fourth root.

&c. &c. &c.

30. If the cube root of the number 10, be extracted, and then the cube root of this root, and of each successive root, the indices of these roots, will be the successive powers of ⅓, the index of the cube root (see art. 12). Thus, by the common rule for extracting the cube root, we have,

Cube root of 10.00000 = 2.15443, index $\frac{1}{3}$

Cube root of 2.15443 = 1.29155, index $\frac{1}{9}$

Cube root of 1.29155 = 1.08902, index $\frac{1}{27}$

Cube root of 1.08902 = 1.02883, index, $\frac{1}{81}$

&c. &c. &c. &c.

On this principle, the following table is constructed:

Table of Odd Roots.

$10^{\frac{1}{3}} = 2.15443$ The cube root.

$10^{\frac{1}{9}} = 1.29155$ The ninth root.

$10^{\frac{1}{27}} = 1.08902$ The 27th root.

$10^{\frac{1}{81}} = 1.02883$ The 81th root.

$10^{\frac{1}{243}} = 1.00952$ The 243d root.

$10^{\frac{1}{729}} = 1.00316$ The 729th root.

&c. &c. &c.

31. The *roots*, or *fractional powers* of 10, in the two preceding tables, are *natural numbers*, and their *indices* the *logarithms* of those numbers (see art. 16). Hence, the construction of the following Skeleton Table composed of two parts, is thus rendered manifest. For, Part I. is deduced from the *Table of Even Roots*, extended by means of eighteen successive extractions of the square root, as directed in art. 28; the left hand column containing the roots or

numbers thus obtained, and the right hand column the decimals equivalent to the fractional indices of those roots or numbers. In like manner, Part II. is deduced from the *Table of Odd Roots*, extended by means of eleven extractions of the cube root, as directed in art. 30; the left hand column containing the roots or numbers thus obtained, and the right hand column the decimals equivalent to the fractional indices of those roots or numbers:

Third Skeleton Table of Logarithms

PART I.

Natural Numbers.	Logarithms.
3.16228	.500000
1.77828	.250000
1.33352	.125000
1.15478	.062500
1.07461	.031250
1.03663	.015625
1.01815	.007813
1.00904	.003906
1.00451	.001953
1.00225	.000977
1.00113	.000488
1.00056	.000244
1.00028	.000122
1.00014	.000061
1.00007	.000031
1.00004	.000015
1 00002	.000008
1.00001	.000004
&c.	&c.

PART II.

Natural Numbers.	Logarithms.
2.15443	.333333
1.29155	.111111
1.08902	.037037
1.02883	.012346
1.00953	.004115
1.00316	.001372
1.00105	.000457
1.00035	.000152
1.00012	.000051
1.00004	.000017
1.00001	.000006
&c.	&c.

32. By means of the three Skeleton tables, and the principles already explained, the logarithms of all natural numbers may be found to any extent required.

33. *To find the logarithm of any prime number.* Rule 1. Divide the given prime number by the natural number

nearest to it, in the Skeleton Tables, but less; divide the quotient by the natural number nearest to it, but less: divide this quotient by the natural number nearest to it, but less; and so on, till the last quotient coincide with some natural number in the tables; then, the last quotient with all the divisors are the tabular factors of which the prime number is composed. Consequently, if the logarithms of all these factors, given in the tables, be added together, their sum will be the logarithm of the given prime number. On this principle, the following tablet, exhibiting the method of calculating the logarithm of the prime number 2, is constructed:

Dividends.		*Divisors.*		*Quotients.*		*Logs. of Divisors.*
2.00000	÷	1.77828	=	1.12468	.	.250000
1.12468	÷	1.07461	=	1.04660	.	.031250
1.04660	÷	1.03663	=	1.00961	.	.015625
1.00961	÷	1.00904	=	1.00057	.	.003906
1.00057	÷	1.00056	=	1.00001	.	.000244
1.00001	÷	1.00001	=	1.00000	.	.000004
				Logarithm of 2 = Sum		.301029

34. *To find the logarithm of any prime number.* Rule 2. Look for the tabular number nearest to the given prime number, but greater; divide the former by the latter; divide the quotient by the tabular number nearest to it, but less; and so on, as before, till the last quotient coincide with some tabular number; then, the last quotient with all the divisors but the first, are the tabular factors of the first quotient. Consequently, if the sum of the logarithms of these factors, which is the logarithm of the first quotient, be subtracted from the logarithm of the first dividend, the remainder will be the logarithm of the given prime number. On this principle, the following tablet exhibiting another method of calculating the logarithm of 2, is constructed:

Dividends.		*Divisors.*		*Quotients.*		*Logs. of Divisors.*
2.15443	÷	2.00000	=	1.07722	.	.333333
1.07772	÷	1.07461	=	1.00243	.	.031250
1.00243	÷	1.00225	=	1.00018	.	.000977
1.00018	÷	1.00014	=	1.00004	.	.000061
1.00004	÷	1.00004	=	1.00000	.	.000015
				Sum of the logarithms of the factors		.032303
				Logarithm of 2 = Remainder		.301030

The latter logarithm of 2 is more correct than the former, owing to the difference in the mode of calculation. The logarithm of 2 calculated to 10 places of decimals, is .3010299957.

35. As the prime number 5 is the quotient of 10 divided by 2, its logarithm is found on the principle that if the logarithm of the dividend be subtracted from the logarithm of the divisor, the remainder is the logarithm of the quotient (see art. 19). Hence the reason of the following calculation is evident:

$$\begin{array}{rl} \textit{Logarithm of } 10 = & 1.000000 \\ \textit{Logarithm of } 2 = & .301030 \\ \hline \textit{Logarithm of } 5 = & .698970 \end{array}$$

36. By the application of either of the preceding methods, or by a judicious combination of both, the logarithms of all the prime numbers to any extent may be found. The following tablet exhibits the logarithms of some prime numbers, which may be calculated in the manner proposed:

Logarithms of Prime Numbers.

Natural Numbers.	Logarithms.
3	.477121
7	.845098
11	1.041393
13	1.113943
17	1.230449
19	1.278754
23	1.361728
29	1.462398
31	1.491362
37	1.568202
41	1.612784
101	2.004321
1013	3.005609
&c.	&c.

37. The logarithms of the powers of a prime number are found by multiplying its logarithm by the indices of those powers (see art. 21). On this principle, the following tablets are constructed:

Logarithms of the Powers of 2.

Log. $4 = 2 \times .301030 = .602060$
Log. $8 = 3 \times .301030 = .903090$
Log. $16 = 4 \times .301030 = 1.204120$
Log. $32 = 5 \times .301030 = 1.505150$
Log. $64 = 6 \times .301030 = 1.806180$
&c. &c. &c.

Logarithms of the Powers of 3.

$$\text{Log. } 9 = 2 \times .477121 = .954243$$
$$\text{Log. } 27 = 3 \times .477121 = 1.431364$$
$$\text{Log. } 81 = 4 \times .477121 = 1.908485$$
$$\text{Log. } 243 = 5 \times .477121 = 2.385606$$
$$\text{Log. } 729 = 6 \times .477121 = 2.862728$$

&c. &c. &c.

38. The logarithms of the composite numbers are found by the addition of the logarithms of the factors (see art. 18.) On this principle the following tablet is constructed:

Logarithms of Composite Numbers.

$$\text{Log. } 6 = \text{log. } 2 + \text{log. } 3 = .778151$$
$$\text{Log. } 12 = \text{log } 2 + \text{log. } 6 = 1.079181$$
$$\text{Log. } 18 = \text{log. } 3 + \text{log. } 6 = 1.255273$$

&c. &c. &c.

$$\text{Log. } 14 = \text{log. } 2 + \text{log. } 7 = 1.146128$$
$$\text{Log. } 21 = \text{log. } 3 + \text{log. } 7 = 1.322219$$
$$\text{Log. } 28 = \text{log. } 4 + \text{log. } 7 = 1.447158$$

&c. &c. &c.

$$\text{Log. } 15 = \text{log. } 3 + \text{log. } 5 = 1.176091$$
$$\text{Log. } 20 = \text{log. } 2 + \text{log. } 10 = 1.301030$$
$$\text{Log. } 25 = \text{log. } 5 + \text{log. } 5 = 1.397940$$

&c. &c. &c.

$$\text{Log. } 105 = \text{log. } 3 + \text{log. } 5 + \text{log. } 7 = 2.021189$$
$$\text{Log. } 385 = \text{log. } 5 + \text{log. } 7 + \text{log. } 11 = 2.585461$$
$$\text{Log. } 1001 = \text{log. } 7 + \text{log. } 11 + \text{log. } 13 = 3.000434$$

&c. &c. &c.

39. The integer prefixed to the decimal part of a logarithm is called its *index* or *characteristic.* Thus, in the preceding table, the logarithm of 20 is 1.301030, of which 1 is the index or characteristic, and .301030, is the decimal part

40. From the Skeleton tables and the preceding articles, it is evident, *First*, that the index of the logarithm of every number of which the highest place is units, is 0; the index of the logarithm of every number of which the highest place is tens, is 1; the index of the logarithm of every number of which the highest place is hundreds, is 2; thousands, 3; and so on. Hence, generally, *the index of the logarithm of every integer is a number less by unity than that which denotes its highest place.* The index of the logarithm of a mixed number being determined solely by its highest integer place, is, of course, not affected by the decimal.

41. *Secondly*, The index of the logarithm of every decimal of which the highest place is tenths, is—1; the index of the logarithm of every decimal of which the highest place is hundredths, is —2; thousandths, —3; and so on. Hence, generally, *the index of the logarithm of every decimal, is a number denoting its highest place, with the negative sign attached to it.* The use of this sign, which is usually written *above* the index, is to indicate that when the logarithm of a decimal is added, its index is to be subtracted, and when the logarithm of a decimal is subtracted, its index is to be added.

42. In Tables of logarithms, only the decimal parts of the logarithms of the natural numbers, are printed; hence, the preceding rules for supplying their indices, are indispensably necessary for the purposes of calculation. To facilitate this process, however, the following table is likewise subjoined:

TABLE OF INDICES OF LOGARITHMS.

PART I.

For Integers.	*Indices.*
Units . . .	0
Tens . . .	1
Hundreds . . .	2
Thousands . . .	3
Tens of Thousands . . .	4
Hundreds of Thousands . . .	5
Millions . . .	6
Tens of Millions . . .	7
Hundreds of Millions . . .	8
Thousands of Millions . . .	9
Tens of Thousands of Millions . . .	10
&c.	&c.

PART II.

For Decimals.	*Indices.*
Tenths . .	$\bar{1}$
Hundredths . .	$\bar{2}$
Thousandths . .	$\bar{3}$
Tenths of Thousandths . .	$\bar{4}$
Hundredths of Thousandths . .	$\bar{5}$
Millionths . .	$\bar{6}$
Tenths of Millionths . .	$\bar{7}$
Hundredths of Millionths . .	$\bar{8}$
Thousandths of Millionths . .	$\bar{9}$
Tenths of Thousandths of Millionths . .	$\overline{10}$
&c.	&c.

43. As an additional illustration of the principles on which the indices of logarithms are supplied, the following table is subjoined, exhibiting the change that takes place on

the index of the logarithm of a number, by merely lowering its value in the decimal scale of notation:

Number.	*Logarithms.*
100100	5.000434
10010	4.000434
1001	3.000434
100.1	2.000434
10.01	1.000434
1.001	0.000434
.1001	$\bar{1}$.000434
.01001	$\bar{2}$.000434
.001001	$\bar{3}$.000434
.0001001	$\bar{4}$.000434
.00001001	$\bar{5}$.000434
&c.	&c.

44. The preceding tables and remarks clearly show the advantages, over every other, which the common system of logarithms possesses, in consequence of its base being the same as the root of the decimal scale of notation. By merely increasing or diminishing, by unity, the index of the logarithm of a number, the logarithm of a decimal multiple or submultiple of that number,is immediately obtained. Hence, the calculation of the logarithm of one number is sufficient for the determination of innumerable others; for, by tabulating the decimal parts of the logarithms of all integers from 1 to 10,000, or from 1 to 100,000, &c., the complete logarithms of such numbers can easily be found, whether they be considered as integers, decimals, or mixed numbers; the proper indices being supplied according to the foregoing rules.

45. A system of logarithms founded on any other base but 10, would want all the advantages above-mentioned. The logarithms of all such numbers as are determined by the mere change of the index in the common system, would require to be separately calculated and tabulated with their indices. The logarithms of all fractions as well as integers, and the logarithms of all numbers of which the factors were powers of the base, would require the same operation to be performed. For, though in the latter case, the calculation of the logarithms would be as easy as before, yet their tabulation with indices would still be necessary, as the bare inspection of the numbers themselves, would not be sufficient to suggest the proper change to be made on the indices, as in the common system. The disadvantages of such a system, would even be more strongly felt in the reverse operation of finding from the tables the numbers corresponding to any given logarithms.

46. In addition to the decimal parts of the logarithms of the common system, which are given in TABLES of loga-

rithms, the average differences of every five logarithms are given in an adjoining column, for the purpose of rendering it easy to obtain the approximate logarithms of numbers greater than those contained in this table. The approximate logarithms of such numbers are obtained on the principle, that the differences of numbers which differ *little* from each other, are *nearly* proportional to the differences of their logarithms. Thus, in Part I. of the Skeleton Table, art. 31, the successive differences of the numbers 1.00056, 1.00028, and 1.00014, are .00028 and .00014; and the differences of their logarithms are .000122 and .00061; now, the following proportion is correct, as far as the decimals extend:

$$.00028 : .00014 :: .000122 : .00061.$$

But were the decimals farther extended, this proportion would be found to be only *nearly* correct. The application of the principle thus established, however, is sufficiently correct for all practical purposes.

NEPERIAN SYSTEM.

47. The system of Logarithms first invented by *Napier*, and sometimes, but improperly denominated the *Hyperbolic*, is, theoretically speaking, the most natural. The base of this system, which is easily deduced from an analytical formula called the *Exponential Theorem*, is 2.718281828459 &c.; this number, however, can only be accurately expressed by the following infinite series;

$$1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\text{\&c.}$$

which may be otherwise expressed thus:

$$2+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\text{\&c.}$$

48. The mathematical construction of logarithms depends on an analytical formula, denominated the *Logarithmic Series*, in which it is shown that the logarithm of a number in every system, can be expressed by the same infinite series, united to a factor called the *Modulus*, which is a constant function, or invariable modification of the base.

49. In the *Logarithmic Series*, the *modulus* is such a function of the *base*, that if an *integer* be assumed as the base of a system, the *modulus* of that system becomes an infinite series, as in the *common* system; and if an *integer* be assumed as the *modulus* the *base* becomes an infinite series, as in the *Neperian* system.

50. The peculiarity which distinguishes the *Neperian system* of logarithms from every other, consists in the simplicity of its *modulus*, which is *unity*. By the adoption of this *modulus*, the logarithms in this system are evidently rendered independent of the base; hence, it is called the most natural. This remark shows that it was possible for *Napier*, the original inventor, to construct his logarithms

without reference to any assumed number as a base. The wonder still is, how he made the discovery half a century before the *Logarithmic Series* was known.

51. From art. 48, it is easily seen that the logarithms of the same number in different systems, are proportional to the *moduli* of those systems respectively. Hence, the *modulus* of the *Neperian* system being *unity*, the *modulus* of the *common* system is found by the following proportion: As *Neperian logarithm of* 10 : *common logarithm of* 10 : : *modulus of Neperian system* : *modulus of common system;* but common logarithm of 10 $=1$; therefore, by the rule of Proportion, the fourth term is

$$\frac{1}{Nep.\ log.\ 10} = mod.\ com.\ system;$$

that is, the *modulus* of the *common* system of logarithms is the *reciprocal* of the *Neperian* logarithm of its base.

52 The *Logarithmic Series* is analytically exhibited in a variety of curious forms. The following rule, which is a verbal translation of one of the most useful of these forms, may be employed in the construction of a Table, either of *Neperian* or of *common* logarithms. It is universally applicable, and possesses this valuable property, that the infinite series converges with greater rapidity, in proportion as the given number increases in magnitude.

53. *To find the Neperian, and thence the common logarithm of a given number*, the Neperian logarithm of the difference between that number and unity being given. *Rule:* Divide unity by the *difference* between double the given number and unity, for a first quotient; divide this quotient by the *square* of *that difference*, for a second quotient; divide the second quotient by the *same square*, for a third quotient; divide the third quotient, by the same square, for a fourth quotient; and so on. Divide these quotients respectively by the odd numbers in the series 1, 3, 5, 7, 9, 11, 13, &c.; that is, divide the first quotient, by 1; the second, by 3; the third, by 5; the fourth, by 7; and so on. Find the sum of as many of the latter quotients as have significant figures two or three decimal places beyond the extent to which the logarithms are required to be accurate; then, to double this sum, add the *Neperian* logarithm of the difference between the given number and unity, and the result is the *Neperian* logarithm of the given number. Lastly, multiply this logarithm by the *modulus* of the common system of logarithms, and the product will be the *common logarithm* of the given number.

54. Example 1. To find the *Neperian* logarithm of the number 2. Subtract unity from 4, which is double the given number, and divide unity by the remainder 3; then, divide this quotient by the square of 3, which is 9, and so on, as in the following operation:

Dividends.	*Quotients.*	
1.00000000 $\div$ 3	= .33333333	First
.33333333 $\div$ 9	= .03703704	Second
.03703704 $\div$ 9	= .00411523	Third
.00411523 $\div$ 9	= .00045725	Fourth
.00045725 $\div$ 9	= .00005081	Fifth
.00005081 $\div$ 9	= .00000565	Sixth
.00000565 $\div$ 9	= .00000063	Seventh

Now, divide these quotients respectively by the series of odd numbers, beginning at unity, as follows:

Dividends.	*Quotients.*
.33333333 $\div$ 1	= .33333333
.03703704 $\div$ 3	= .01234568
.00411523 $\div$ 5	= .00082305
.00045725 $\div$ 7	= .00006532
.00005081 $\div$ 9	= .00000565
.00000565 $\div$ 11	= .00000051
.00000051 $\div$ 13	= .00000005
	Sum .34657359

To double this sum, which is .69314718, add the *Neperian* logarithm of 1, the difference between the given number and unity; in this case, the result will still be .69314718; for, the logarithm of 1 is 0, in the *Neperian*, as in every other system. Therefore, .69314718 is the *Neperian* logarithm of the number 2.

55. Example 2. To find the *Neperian* logarithm of the number 4, which is the square of 2. Multiply .69314718 by 2, the index of the square, and the product 1.38629436 is the *Neperian* logarithm of 4 (art. 21).

56. Example 3. To find the *Neperian* logarithm of 5. Divide unity by 9, the difference between double 5 and unity; then divide the quotient by 81, the square of 9, and so on, as follows:

Dividends.	*Quotients.*
1.000000000 $\div$ 9	= .111111111
.111111111 $\div$ 81	= .001371742
.001371742 $\div$ 81	= .000016935
.000016935 $\div$ 81	= .000000209

These quotients are now to be divided by the series of odd numbers, as follows:

Dividends.		*Quotients.*
.111111111	$\div$ 1 =	.111111111
.001371742	$\div$ 3 =	.000457247
.000016935	$\div$ 5 =	.000003387
.000000209	$\div$ 7 =	.000000080
	Sum	.111571775

To double this sum, which is .22314355, add 1.88629436 the *Neperian* logarithm of 4, the difference between the given number and unity, and the result 1.60943791 is the *Neperian* logarithm of 5.

57. Example 4. To find the *Neperian* logarithm of 10, add together the *Neperian* logarithms of 2 and 5, and the sum 2.30258509 is the *Neperian* logarithm of 10 (art. 18). Consequently, the reciprocal of this number which is .4342944819, is the *modulus* of the *common* system of logarithms.

58. Example 5. To find the *common* logarithms of 2, 4, and 5. Multiply the *Neperian* logarithms of these numbers by the *modulus* of the *common* system, and the products will be the common logarithms of the numbers, as follows:

Numbers.	*Neperian logs.*		*Modulus.*	*Common logs.*
2	. 0.69314718	$\times$	.4342944819 =	0.301030
4	. 1.38629436	$\times$	.4342944819 =	0.602060
5	. 1.60943791	$\times$	.4342944819 =	0.698970

59. The process of multiplication indicated above may be reduced to that of simple addition, by employing *Table* A., which, as well as *Table* V., will be found very useful in the construction and conversion of logarithms.

60. The construction of *Tables of Natural Sines, Cosines, Tangents, &c.* being explained in almost every recent treatise on Trigonometry, it will be sufficient to remark here, that the *Logarithmic Tables* III. *and* IV., are constructed either by finding the logarithms of the *natural numbers* inserted in those tables, or by employing several ingenious analytical formulæ, of which the most noted are:

$\text{Log.Sin.}(90^\circ \times m:n) = \text{Log.}m + \text{Log.}(2n-m) + \text{Log.}(2n+m) - 3\,\text{Log.}\,n + 9.5940598857 - [0.0700228266(m:n)^2 + 0.0011172664\,(m:n)^4 + 0.0000392291\,(m:n)^6 + 0.0000017293\,(m:n)^8 + 0.0000000844(m:n)^{10} + 0.0000000043(m:n)^{12} + \&c.]$.

$\text{Log.Cos.}(90^\circ \times m:n) = \text{Log.}(n-m) + \text{Log.}(n+m) - 2\,\text{Log.}\,n + 10 - [0.1014948593\,(m:n)^2 + 0.0031872941\,(m:n)^4 + 0.0002094858\,(m:n)^6 + 0.0000168483\,(m:n)^8 + 0.0000014802\,(m:n)^{10} + 0.0000001365(m:n)^{12} + \&c.]$.

THE

UNIVERSAL CALCULATOR'S

POCKET GUIDE.

SECTION I.—INVESTIGATION OF THE FORMULÆ.

61. IN the preceding introduction, the construction of logarithms has been explained on arithmetical principles; we now proceed to explain the subject by algebraic formulæ. Let $a^x = N$, then $x = \log N$, a being the base of the system. The expression a^x may be put under the form $(1 + a - 1)^x$, that is, $a^x = (1 + a - 1)^x$; now, for the sake of convenience, put $a - 1 = b$; we have then, by the Binomial Theorem, the following development:

$$(1 + b)^x = 1 + xb + \frac{x(x-1)}{1.2}b^2 + \frac{x(x-1)(x-2)}{1.2.3}b^3 + \&c.$$

By multiplying together the terms of each of the numerators of the coefficients of the powers of b, we have

$$(1 + b)^x = 1 + xb + \frac{x^2 - x}{1.2}b^2 + \frac{x^3 - 3x^2 + 2x}{1.2.3}b^3 + \&c.$$

And, by separating the different parts of the coefficients of the powers of b, and arranging all the terms according to the powers of x, we have

$$(1 + b)^x = 1 + x\left(b - \frac{b^2}{2} + \frac{b^3}{3} - \&c.\right) + \frac{x^2}{1.2}$$

$$(b^2 - b^3 + \&c.) + \frac{x^3}{1.2.3}(b^3 - \&c.) + \&c.$$

62. Now, if the coefficients and terms of the last form of the development of $(1 + b)^x$ be extended a little further, and if the portions of the coefficients of the successive powers of x, which are contained within the brackets, be each divided by that which immediately precedes it, we shall find that

the quotients will be always the same, namely, the coefficient of the first power of x; whence, it follows, that these successive coefficients are the ascending and positive powers of the first coefficient or that of x. Let this coefficient be denoted by p, that is, let

$$b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\frac{b^5}{5}+\text{\&c.}=p;$$

then, the expansion of $(1+b)^x$ or $(1+a-1)^x$, will assume the form of

$$a^x=1+px+\frac{p^2x^2}{1.2}+\frac{p^3x^3}{1.2.3}+\frac{p^4x^4}{1.2.3.4}+\text{\&c. (I.)}$$

This formula is called the *Exponential Theorem*, see art. 47.

63 Again, let the expression, N^x be put under the form $(1+N-1)^x$; and, put $N-1=d$; then, by reasoning in in the same manner as above, we shall have

$$(1+d)^x=1+x(d-\frac{d^2}{2}+\frac{d^3}{3}-\text{\&c.})+\frac{x^2}{1.2}(d^2-d^3$$

$$+\text{\&c.})+\text{\&c.}$$ And, putting $d-\frac{d^2}{2}+\frac{d^3}{3}-\text{\&c.}=q$,

we have $N^x=1+qx+\frac{q^2x^2}{1.2}+\frac{q^3x^3}{1.2.3}+\frac{q^4x^4}{1.2.3.4}+$ &c. In this formula, making $x=1$, we have

$$N=1+q+\frac{q^2}{1.2}+\frac{q^3}{1.2.3}+\frac{q^4}{1.2.3.4}+\text{\&c. (II.)}$$

64. Now, since $a^x=N$, the developments of both members of this equation must be equal, that is, the right hand members of the equations marked I. and II.; whence we have

$$1+px+\frac{p^2x^2}{1.2}+\text{\&c.}=1+q+\frac{q^2}{1.2}+\text{\&c.},$$

and consequently,

$$(1-1)+(px-q)+\frac{1}{1.2}(p^2x^2-q^2)+\text{\&c.}=0;$$

In order to satisfy this equation, we must have $px-q=0$, and therefore $px=q$; this would also appear by the theory of indeterminate coefficients; whence we have $x=\frac{q}{p}$, and consequently,

$$\log N = \frac{d - \frac{d^2}{2} + \frac{d^3}{3} - \frac{d^4}{4} + \&c.}{b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \&c.},$$

or, by substituting the values of d and b, we have

$$\log N = \frac{(N-1) - \frac{1}{2}(N-1)^2 + \frac{1}{3}(N-1)^3 - \&c.}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.} \text{(III.)}$$

65. Now, as N denotes any number whatever, we may add unity to it, and we shall have

$$\log (1 + N) = \frac{N - \frac{1}{2} N^2 + \frac{1}{3} N^3 - \&c.}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.}$$

As the denominator of the right hand member of this equation is a function of the base a, we shall denote its reciprocal by M; this reciprocal indeed is what is called the *modulus*, (see art. 48.) because it modulates or regulates the logarithm of a number, according to the *base*, or value of a, that may be assumed. We shall then have

$$M = \frac{1}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \&c.}, \text{ and}$$

$$\log (1 + N) = M \left\{ N - \tfrac{1}{2} N^2 + \tfrac{1}{3} N^3 - \&c. \right\} \text{ (IV.)}$$

66. If in formula I., art. 62, we make $x = \frac{1}{p}$, we shall then have

$$a^{\frac{1}{p}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.$$

The summation of this series arithmetically to a certain number of terms, gives the number 2.718281828459 &c., which is commonly denoted by the Greek letter ε. Now, if $p = 1$, then $a = \varepsilon$, and $b = a - 1$ (art. 61.) becomes $\varepsilon - 1$; moreover, since $(b - \frac{1}{2} b^2 + \frac{1}{3} b^3 - \&c.) = p$, (art. 62.) we shall then have $(\varepsilon - 1) - \frac{1}{2}(\varepsilon - 1)^2 + \frac{1}{3}(\varepsilon - 1)^3 - \&c. = 1$. But since $a^{\frac{1}{p}} = \varepsilon$, it follows, by involution, that $a = \varepsilon^p$, and $a^{-1} = \varepsilon^{-p}$, or $\frac{1}{a} = \varepsilon^{-p}$; whence, $-p = \log \frac{1}{a}$ to the base ε, and consequently, by formula III., we have

$$\log_{\epsilon} \frac{1}{a} = -p = \frac{\left(\frac{1}{a}-1\right) - \frac{1}{2}\left(\frac{1}{a}-1\right)^2 + \frac{1}{3}\left(\frac{1}{a}-1\right)^3 - \text{\&c.}}{(\epsilon - 1) - \frac{1}{2}(\epsilon - 1)^2 + \frac{1}{3}(\epsilon - 1)^3 - \text{\&c.}}$$

Again, since the denominator of the right hand member of this equation has been shown to be equal to unity, we have, by changing signs in both members, and reducing, the formula

$$+p = \left(\frac{a-1}{a}\right) + \frac{1}{2}\left(\frac{a-1}{a}\right)^2 + \frac{1}{3}\left(\frac{a-1}{a}\right)^3 + \text{\&c.}$$

in which all the terms are positive, and the series is converging. If the value of a in this series be assumed equal to 10, the base of the common system of logarithms, we shall have, by the arithmetical summation of a certain number of terms, $p = 2.302585092994$ &c.

67. Again, since $a = \epsilon^p$, we have $p = \log a$ to the base ϵ; and, consequently,

$$\log_{\epsilon} a = p = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \text{\&c.}$$

But, since $M = \frac{1}{p}$ (art. 64.) therefore,

$$M = \frac{1}{2.302585092994 \text{ \&c.}} = .434294481903 \text{ \&c.}$$

This number is the modulus of the common system of logarithms. See art. 51 and 57.

68. If in formula III., art. 64, we make $N = 1 - N$, and substitute M for its value in the right hand member, we shall then have

$$\log (1 - N) = M \left\{ -N - \tfrac{1}{2}N^2 - \tfrac{1}{3}N^3 - \text{\&c.} \right\} \quad \text{(V.)}$$

Now, by subtracting the right hand member of this equation from that of equation IV., and remembering the principle explained in art. 19, we shall have

$$\log \frac{1+N}{1-N} = 2M \left\{ N + \tfrac{1}{3}N^3 + \tfrac{1}{5}N^5 + \text{\&c.} \right\} \quad \text{(VI.)}$$

69. Again, if in the latter formula, we substitute n for $\frac{1+N}{1-N}$, we shall obtain a formula more convenient for the computation of logarithms. For, since by hypothesis,

$$n = \frac{1+N}{1-N}, \text{ we obtain } N = \frac{n-1}{n+1},$$

by the resolution of the former of these two equations; and by the substitution of the value of N, in formula VI., we have formula VII., viz:

$$\log n = 2\,\mathrm{M}\left\{\left(\frac{n-1}{n+1}\right) + \tfrac{1}{3}\left(\frac{n-1}{n+1}\right)^3 + \tfrac{1}{5}\left(\frac{n-1}{n+1}\right)^5 + \&c.\right\}$$

If in this equation, we make $n = 2$, we shall have

$$\log 2 = .43429448\ \&c.\ \left\{\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{3.3^5} + \frac{1}{3.3^7} + \&c.\right\}$$

and consequently, $\log 2 = .301030$ &c.

70. Again, if in formula VI., we put $\dfrac{1+\mathrm{N}}{1-\mathrm{N}} = \dfrac{n+m}{m}$, from which we find $\mathrm{N} = \dfrac{m}{2n+m}$, we shall have, by substitution, &c., formula VIII., viz.: $\log(n+m) = \log n +$

$$2\mathrm{M}\left\{\frac{m}{2n+m} + \tfrac{1}{3}\left(\frac{m}{2n+m}\right)^3 + \tfrac{1}{5}\left(\frac{m}{2n+m}\right)^5 + \&c.\right\}$$

And, if in this formula, we make $m = 1$, we shall have formula IX., viz.: $\log(n+1) = \log n +$

$$2\mathrm{M}\left\{\frac{1}{2n+1} + \tfrac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \tfrac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \&c.\right\}$$

71. Next, if in the latter formula, we make $n = \mathrm{N} - 1$, we shall have formula X., viz.: $\log \mathrm{N} = \log(\mathrm{N}-1) +$

$$2\mathrm{M}\left\{\frac{1}{2\mathrm{N}-1} + \tfrac{1}{3}\left(\frac{1}{2\mathrm{N}-1}\right)^3 + \tfrac{1}{5}\left(\frac{1}{2\mathrm{N}-1}\right)^5 + \&c.\right\}$$

If, in this formula, we take $\mathrm{M} = 1$, the series will give the *Neperian* logarithm of the number N, art. 51. This substitution will give the series referred to in arts. 52 and 53. The advantage of this formula is, that when N is a prime number, $\mathrm{N}-1$ can always be resolved into factors; and consequently, the logarithm of N is more easily computed from this formula, than from any of the preceding.

72. A very rapidly converging series may be obtained from formula X., by substituting in it n^2 for N, from which we find $\mathrm{N} - 1 = n^2 - 1 = (n+1)(n-1)$, and $\log \mathrm{N}^2 = 2 \log \mathrm{N}$; whence by substitution and dividing by 2, we have formula XI., viz.:

$$\log n = \tfrac{1}{2}\left\{\log(n+1) + \log(n-1)\right\} +$$

$$\mathrm{M}\left\{\frac{1}{2n^2-1} + \tfrac{1}{3}\left(\frac{1}{2n^2-1}\right) + \tfrac{1}{5}\left(\frac{1}{2n^2-1}\right)^5 + \&c.\right\}$$

73. Lastly, if in formula VI., we put $\frac{1+N}{1-N} = \frac{(n+2)}{(n-2)} \times \frac{(n-1)^2}{(n+1)^2}$, we shall find $N = \frac{2}{n^3 - 3n}$; and since

$$\log \frac{(n+2)(n-1)^2}{(n-2)(n+1)^2} = \log(n+2) - \log(n-2) + 2\log(n-1) - 2\log(n+1),$$

we shall have by transposition and substitution, formula XII., viz.: $\log(n+2)$

$$= \log(n-2) + 2\log(n+1) - 2\log(n-1) + 2M \times \left\{ \frac{2}{n^3-3n} + \tfrac{1}{3}\left(\frac{2}{n^3-3n}\right)^3 + \tfrac{1}{5}\left(\frac{2}{n^3-3n}\right)^5 + \&c. \right\}$$

Logarithmic formulæ of greater convergency than these might be deduced in a similar manner, but the method having been sufficiently exemplified, we must pass to the consideration of the formulæ for sines, cosines, &c.

74. Trigonometric formulæ are most easily investigated by means of the differential calculus. Thus, let $y = \tan x$, and let x become $x + h$, then $y + k = \tan(x + h)$, k being the increment of the function y, and h the increment of the independent variable x. Now, by trigonometry and actual division, we have

$$\tan(x+h) = \frac{\tan x + \tan h}{1 - \tan x \tan h} = \tan x + (\tan^2 x + 1)\tan h + (\tan^2 x + 1)\tan x \tan^2 h + \&c.$$

75. Now, when h is very small, $\tan h = h$ nearly, and we have

$$k = (\tan^2 x + 1)\,h + (\tan^2 x + 1)\tan x . h^2 + \&c.$$

Whence, $\frac{dy}{dx} = \tan^2 x + 1 = \sec^2 x = \frac{1}{\cos^2 x}$, and

$d(\tan x) = \frac{dx}{\cos^2 x}$. But, since $\tan x = \frac{\sin x}{\cos x}$,

and $d\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\, d(\sin x) - \sin x\, d(\cos x)}{\cos^2 x}$;

therefore, $\cos x\, d(\sin x) - \sin x\, d(\cos x) = dx$; now

$$d(\cos x) = d\sqrt{(1-\sin^2 x)} = -\frac{\sin x\, d(\sin x)}{\cos x}:$$

whence, $\cos x\, d(\sin x) + \frac{\sin^2 x}{\cos x} d(\sin x) = dx$; and

since, $\cos x + \frac{\sin^2 x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos x} = \frac{1}{\cos x}$,

therefore, $\frac{d(\sin x)}{\cos x} = dx$, and $d(\sin x) = \cos x\, dx$.

76. From these formulæ, it is very easy to find, that $d(\cos x) = -\sin x\, dx$; $d(\text{versin}\, x) = \sin x\, dx$; $d(\text{coversin}\, x) = -\cos x\, dx$; $d(\cot x) = -(1+\cot^2 x)\, dx = -\text{cosec}^2 x\, dx = -\frac{dx}{\sin^2 x}$; $d(\sec x) = \sec x \tan x\, dx = \sec x\sqrt{(\sec^2 x - 1)}\, dx = \frac{\sin x\, dx}{\cos^2 x}$; and $d(\text{cosec}\, x) = -\text{cosec}\, x \cot x\, dx = -\text{cosec}\, x \sqrt{(\text{cosec}^2 x - 1)}\, dx = \frac{\cos x\, dx}{\sin^2 x}$.

77. To find the series for the sine and cosine in terms of the *length of the arc*, we may now employ Taylor's theorem, which is,

$$f(x+h) = fx + \frac{dfx}{dx}h + \frac{d^2 fx}{dx^2}\frac{h^2}{1.2} + \frac{d^3 fx}{dx^3}\frac{h^3}{1.2.3} + \text{\&c.}$$

where fx denotes any function of x, and $d^2 fx$, $d^3 fx$, &c., the second, third, &c., differentials of fx.

78. Let $fx = \sin x$, then $f(x+h) = \sin(x+h)$; whence, $dfx = d(\sin x) = \cos x dx$, and $\frac{dfx}{dx} = \cos x$; again, $\frac{d^2 fx}{dx} = -\sin x dx$, and $\frac{d^2 fx}{dx^2} = -\sin x$; next, $\frac{d^3 fx}{dx^3} = -\cos x$, $\frac{d^4 fx}{dx^4} = \sin x$, &c. Wherefore, by substitution in Taylor's theorem, we have formula I., viz.:

$$\sin(x+h) = \sin x + \cos x . h - \sin x\frac{h^2}{1.2} - \cos x\frac{h^3}{1.2.3} + \text{\&c.}$$

79. In this formula, let $x = 0$; then $\sin x = \sin 0 = 0$, and $\cos x = \cos 0 = 1$; whence, we have formula II., viz.:

$$\sin h = h - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} - \frac{h^7}{1.2.3.4.5.6.7} + \text{\&c.}$$

80. As x may be the increment and h the variable, by putting x for h, and A, B, C, D, &c., for the first, second, third, fourth, &c., terms, we have formula III., viz :

$$\sin x = x - \frac{x^2}{2.3}\,A - \frac{x^2}{4.5}\,B - \frac{x^2}{6.7}\,C - \&c.$$

81. If in formula I., we make $x = \frac{1}{2}\pi$, where $\pi = 180^\circ$, we shall have $\sin(x+h) = \sin(\frac{1}{2}\pi + h) = \cos h$, $\sin x = \sin\frac{1}{2}\pi = 1$, and $\cos x = \cos\frac{1}{2}\pi = 0$; whence we have formula IV., viz.:

$$\cos h = 1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4} - \frac{h^6}{1.2.3.4.5.6} + \&c.$$

82. By making the same changes in this formula as in formula II., we have formula V., viz.:

$$\cos x = 1 - \frac{x^2}{2}\,A - \frac{x^2}{3.4}\,B - \frac{x^2}{5.6}\,C - \&c.$$

This formula might also have been obtained directly from Taylor's theorem, by making $fx = \cos x$.

83. Having thus obtained formulæ for the computation of the sine and cosine of any arc, in terms of the length of the arc, it is unnecessary to investigate formulæ for that of its tangent, secant, &c.; because when the former are computed, the latter may be found by proportion, or by the known relations which subsist between them. Thus, when radius is unity, we have (1) $\cos x \sec x = 1$; (2) $\sin x \operatorname{cosec} x = 1$; (3) $\cos x \tan x = \sin x$; (4) $\sin x \cot x = \cos x$; (5) $\tan x \cot x = 1$; (6) $\sin^2 x + \cos^2 x = 1$; (7) $\tan^2 x + 1 = \sec^2 x$; and (8) $\cot^2 x + 1 = \operatorname{cosec}^2 x$.

84. It will now be necessary to investigate a formula for the determination of the length of an arc, in terms of one of its functions, say the tangent, in order to determine the length of the entire circumference in terms of the radius.

85. Let $fx =$ arc to $\tan x = y$, then $f(x+h) =$ arc to tan $(x+h)$; whence, $dfx = d$(arc to $\tan x$) $= \cos^2 y\,dx$. For, $\tan y = x$, and $d(\tan y) = \frac{dy}{\cos^2 y} = dx$ (art. 75); therefore, $\frac{dy}{dx} = \cos^2 y$; again, $\frac{d^2y}{dx^2} = -2\sin y\cos y\,\frac{dy}{dx} = -\sin 2y\cos^2 y$; next, $\frac{d^3y}{dx^3} = -(\cos 2y\cos^2 y - \sin 2y \sin y\cos y)\,\frac{dy}{dx} = -2\cos 3y$

$\cos y \frac{dy}{dx} = -2\cos 3y \cos^3 y$; in like manner, $\frac{d^4 y}{dx^4} = 2.3 \sin 4y$ $\cos^4 y$; $\frac{d^5 y}{dx^5} = 2.3.4 \cos 5y \cos^5 y$, &c. Wherefore, by substitution in Taylor's Theorem, we have formula I., viz.:

$$\text{arc to tan } (x+h) =$$
$$y + \cos^2 y.h - \tfrac{1}{2}\sin 2y \cos^2 y.h^2 - \tfrac{1}{3}\cos 3y \cos^3 y\, h^3 +$$
$$\tfrac{1}{4}\sin 4y \cos^4 y.h^4 + \text{\&c.}$$

86. In this formula, let $x = 0$, then $y = 0$, $\sin ny = 0$, $\cos y = 1$, and $\cos ny = 1$, where n denotes any number; then we have formula II., viz.:

$$\text{arc to tan}h = h - \tfrac{1}{3}h^3 + \tfrac{1}{5}h^5 - \tfrac{1}{7}h^7 + \text{\&c.}$$

87. Now, if in this formula, we denote arc to $\tan h$ by x, we have formula III., viz.:

$$x = \tan x \quad - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \tfrac{1}{7}\tan^7 x + \text{\&c.} \quad \text{or}$$
$$x = \tan x \left\{ 1 - \tfrac{1}{3}\tan^2 x + \tfrac{1}{5}\tan^4 x - \tfrac{1}{7}\tan^6 x + \text{\&c.} \right\}$$

88. Again, since $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$, and $\tan 4x = \frac{2\tan 2x}{1 - \tan^2 2x}$; if we assume $\tan x = \frac{1}{5}$, we find $\tan 2x = \frac{5}{12}$, and $\tan 4x = \frac{120}{119}$, which is nearly equal to unity, or, $\tan \frac{1}{4}\pi$ to radius 1. Putting, therefore, $h = 4x - \frac{1}{4}\pi$, we find, from the formula $\tan(a-b) = \frac{\tan a - \tan b}{1 - \tan a \tan b}$, (by making $4x = a$, and $\frac{1}{4}\pi = b$,) that $\tan h = \frac{1}{239}$.

89. Whence, by formula III., art. 87, we have

$$4x = 4 \times \tfrac{1}{5}\left\{1 - \tfrac{1}{3}\left(\tfrac{1}{5}\right)^2 + \tfrac{1}{5}\left(\tfrac{1}{5}\right)^4 - \text{\&c.}\right\} = .78959 \text{ \&c}$$
$$h = \tfrac{1}{239}\left\{1 - \tfrac{1}{3}\left(\tfrac{1}{239}\right)^2 + \tfrac{1}{5}\left(\tfrac{1}{239}\right)^4 - \text{\&c.}\right\} = .00419 \text{ \&c.}$$

Consequently, $4x - h = \frac{1}{4}\pi = .7854$, and $\pi = 3.1416$ the length of the circumference of a circle whose diameter is unity, the former number being also its area. These results are obtained by taking only three terms of the value of $4x$, and one of that of h; but by taking 8 terms of the former, and 3 of the latter, we should find $\pi = 3.141592653589793$.

90. Since, $\sin x = x\left(1 - \frac{x^2}{2.3} + \frac{x^4}{2.3.4.5} - \text{\&c.}\right)$

put $\sin x = 0$, then x may be $0, \pi, 2\pi, 3\pi$, &c., or $-\pi, -2\pi, -3\pi$, &c.; and, considering this series as an equation, these values are its roots. Now, by putting $\frac{1}{y}$ for x, and reducing the equation so that the term of the highest dimension (y^n for instance) stands first, its roots will be $\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}$, &c., and $-\frac{1}{\pi}, -\frac{1}{2\pi}, -\frac{1}{3\pi}$, &c. Consequently, $y - \frac{1}{\pi}, y - \frac{1}{2\pi}$, &c., and $y + \frac{1}{\pi}, y + \frac{1}{2\pi}$, &c. are divisors of this equation; whence, by separating it into factors, and dividing by y^n, we have

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right) \text{\&c.}$$

91. Now, by putting $x = \frac{m}{n} \cdot \frac{\pi}{2}$, we have

$$\sin \frac{m}{n} \cdot \frac{\pi}{2} = \frac{m}{n} \cdot \frac{\pi}{2}\left(1 - \frac{m^2}{4n^2}\right)\left(1 - \frac{m^2}{16n^2}\right) \text{\&c.}$$

whence, by taking the logarithms of the factors, we have

$$\log \sin \frac{m}{n} .90^\circ =$$

$$\log \quad + \log m - 3\log n - 3\log 2 + \log(2n + m) + \log (2n - m) + \log\left(1 - \frac{m^2}{4^2 n^2}\right) + \log\left(1 - \frac{m^2}{6^2 n^2}\right) + \text{\&c.}$$

92. Again, since by art. 68, formula V., we have

$$\log\left(1 - \frac{m^2}{4^2n^2}\right) = -\mathrm{M}\left\{\frac{m^2}{4^2n^2} + \frac{m^4}{2.4^4n^4} + \frac{m^6}{3.4^6n^6} + \text{\&c.}\right\}$$

$$\log\left(1 - \frac{m^2}{6^2n^2}\right) = -\mathrm{M}\left\{\frac{m^2}{6^2n^2} + \frac{m^4}{2.6^4n^4} + \frac{m^6}{3.6^6n^6} + \text{\&c.}\right\}$$

&c. &c.

93. Now, summing the coefficients of $\frac{m^2}{n^2}, \frac{m^4}{n^4}$, &c., we obtain the first formula in art. 60, and the second may be obtained in a similar manner. See *Woodhouse's Trig.*, p. 337, et seq.

SECTION II.—APPLICATION OF THE TABLES.

LOGARITHMIC ARITHMETIC.

MULTIPLICATION.

94. *Rule.* Add together the logarithms of the factors, and the sum is the logarithm of the product. *Note.* When negative indices occur, the difference between their sum and that of the *positive* indices, is the index of the product; this index will be either positive or negative, according as the sum of the positive indices is greater or less than the sum of the negative indices. If there be units to carry from the sum of the decimal parts of the logarithms to the sum of their indices, these units, when added, are always to be reckoned positive.

Example 1. Multiply 125 by 256.

$$\begin{aligned} \text{Log. } 125 &= 2.096910 \\ \text{Log. } 256 &= 2.408240 \\ \hline \text{Log. Product } 32000 &= 4.505150 \text{ sum.} \end{aligned}$$

Ex. 2. Find the continuous product of 1.25, .0016, and 18.25.

$$\begin{aligned} \text{Log. } 1.25 &= 0.096910 \\ \text{Log. } .0016 &= \bar{3}.204120 \\ \text{Log. } 18.25 &= 1.261263 \\ \hline \text{Log. Product } .0365 &= \bar{2}.562293 \text{ sum.} \end{aligned}$$

DIVISION.

95. *Rule.* Subtract the logarithm of the divisor from that of the dividend, the remainder is the logarithm of the quotient. *Note.* When the indices of the divisor and dividend are either both negative or both positive, and the index of the dividend greater than that of the divisor, the index of the quotient is their *difference* with the same sign prefixed; but, when the index of the dividend is less than that of the divisor, the contrary sign must be prefixed. When the indices of the divisor and dividend are different, the index of the quotient is their *sum*, with the index of the dividend prefixed. If there be units to carry from the difference of the decimal parts of the logarithms to the difference of their indices, these units, when subtracted, are always reckoned negative.

Ex. 1. Divide 96000 by 768.

$$\begin{aligned} \text{Log. } 96000 &= 4.982271 \\ \text{Log. } 768 &= 2.885361 \\ \hline \text{Log. Quotient } 125 &= 2.096910 \text{ remainder.} \end{aligned}$$

Ex. 2. Divide 75 by 400.

$$\begin{aligned} \text{Log. } 75 &= 1.875061 \\ \text{Log. } 400 &= 2.602060 \\ \hline \text{Log. Quotient } .1875 &= \bar{1}.273001 \text{ remainder.} \end{aligned}$$

Ex. 3. Divide .15625 by .625.

Log. .15625 = $\bar{1}$.193820
Log. .625 = $\bar{1}$.795880

Log. Quotient .25 = $\bar{1}$.397940 remainder.

FRACTIONS.

96. *To find the logarithm of a vulgar fraction, and its decimal equivalent.* Subtract the logarithm of the denominator from that of the numerator, the remainder is the logarithm of the fraction, and the number indicated by it, is the decimal equivalent.

Ex. Find the logarithm and decimal equivalent of $\frac{651}{1736}$.

Log. 651 = 2.813581
Log. 1736 = 3.239550

Log. Decimal .375 = $\bar{1}$.574031 remainder.

97. *To find the logarithm of a mixed number.* Reduce the mixed number to an improper fraction, and find its logarithm as above. Or, find the decimal equivalent of the fraction, annex it to the integer, and find the logarithm as usual.

Ex. Find the logarithm of $9\frac{7}{8}$ or $\frac{79}{8}$.

Log. 79 = 1.897627
Log. 8 = 0.903090

Log. 9.875 = 0.994537

98. *To find the logarithm of the parts of any integer, and their decimal equivalent.* From the logarithm of the given number of parts, reduced, if necessary, into parts of the lowest denomination, subtract the logarithm of the number of such parts in the integer, the remainder is the logarithm required, and the number indicated by it, is the decimal equivalent.

Ex. Find the logarithm and decimal equivalent of 17 sh. $10\frac{1}{2}$d. or 858 farthings.

Log. 858 = 2.933487
Log. 960 = 2.982271

Log. .89375 = $\bar{1}$.951216

PRACTICE.

99. *To find the cost, when the quantity and price are given.* Add the logarithm of the price to the logarithm of the quantity, and the sum is the logarithm of the cost.

Ex. What is the cost of 486 yards at 18 sh. $7\frac{1}{2}$d. per yard.

Log. 486 = 2.686636
Log. .93125 = $\bar{1}$.969067

Log. 452.5875 = 2.655703
Cost 452£ 11sh. 9d. Ans.

100. *To find the price, when the cost and quantity are given.* Subtract the logarithm of the quantity from the logarithm

of the cost, and the remainder is the logarithm of the price.

Ex. If 37 cwts. 2 qrs. cost 131L. 5s. what is the price per lb.

Log. 131.25 = 2.118100
Log. 4200 = 3.623249
Log. .03125 = $\bar{2}$.494851 remainder.
L.03125 = 7½d. per lb. Ans.

101. *To find the quantity when the cost and price are given.* Subtract the logarithm of the price from the logarithm of the cost, and the remainder is the logarithm of the quantity.

Ex. How many yards can be had for 10L. 6s. 3d. at 5s. 6d. per yard.

Log. 10.3125 = 1.013364
Log. .275 = $\bar{1}$.439333
Log. 37.5 = 1.574031 remainder.
37.5 Yards = 37½ yds. Ans.

PROPORTION.

102. *Rule.* From the sum of the logarithms of the second and third terms, subtract the logarithm of the first term, the remainder is the logarithm of the fourth term. *Otherwise.* Add together the logarithms of the second and third terms and the arithmetical complement of the logarithm of the first term, and from the sum subtract 10, the remainder is the logarithm of the fourth term. *Note.* The arithmetical complement is the difference between a logarithm and the number 10, or any multiple of this number.

Ex. If a clerk's salary be L.80 per annum, how much is due for the period from 1st January, till 15th May.

As 365d. : 134d. : : L.80 : 29L. 7s. 4¼d.
For, Log. 134 = 2.127105
Log. 80 = 1.903090
4.030195 sum.
Log. 365 = 2.562293
Log. 29.37 = 1.467902 remainder.
And, L.29.37 = 29L. 7s. 4¼d.

103. The following ready methods of calculating the yearly income from the daily allowance, may be useful to some readers.

I. *When Sundays are included.* Take the daily allowance in pence, and say, "As many pounds, as many half pounds, and as many fivepences added together, make the yearly income."

II. *When Sundays are excluded.* Take the daily allowance in pence, and say, "As many pounds, as many crowns, as many shillings, and as many pence, added together, make the yearly income."

Ex. What is the yearly income at 2s. 6d., or 30d. a-day.

1. *Inclusive of Sundays.*

30 Pounds	=	L.30 : 0 : 0
30 half Pounds	=	15 : 0 : 0
30 Fivepences	=	0 : 12 : 6
Yearly income		L.45 : 12 : 6 Ans.

2. *Exclusive of Sundays.*

30 Pounds	=	L.30 : 0 : 0
30 Crowns	=	7 : 10 : 0
30 Shillings	=	1 : 10 : 0
30 Pence	=	0 : 2 : 6
Yearly income		L.39 : 2 : 6 Ans.

COMPOUND PROPORTION.

104. *Rule.* Add together the logarithms of all the second terms and the logarithm of the third term; then, add together the logarithms of all the first terms, and subtract the latter sum from the former, the remainder is the logarithm of the fourth term. *Otherwise.* Add together the logarithms of all the second terms, the logarithm of the third term, and the arithmetical complements of the logarithms of all the first terms, and from the sum subtract as many tens as the arithmetical complements indicate, the remainder is the logarithm of the fourth term.

Ex. If a puncheon of spirits containing 85 gallons cost 58L. 8s. 9d. what ought to be the cost of a hhd. or 63 gallons of the same spirits, reduced by adding 1 gallon of water to 4 gallons of spirits.

As $\left\{ \begin{matrix} 85g. : 63g. \\ 5g. : 4g. \end{matrix} \right\}$: : 58L. 8s. 9d. : 34L. 13s.

For, Arith. co. log. 85 = 8.070581
Arith. co. log. 5 = 9.301030
Log. 63 = 1.799341
Log. 4 = 0.602060
Log. 58.4375 = 1.766692

Log. 34.65 = 1.539704

And, L.34.65 = 34L. 13s. Ans.

PER CENTAGES,

105. *To find the allowance on a given principal, at a given rate per cent.* Add together the logarithms of the given principal and rate per cent., and from their sum subtract 2, the remainder is the logarithm of the allowance required.

Ex. What is the gain on 470L. 10s., at 30 per cent.

Log. 470.5 = 2.672560
Log. 30 = 1.477121

Log. 141.15 = 2.149681 sum—2.
L.141.15 = 141L. 3s. Ans.

106. *To find the rate per cent. of an allowance on a given*

principal. Add 2 to the logarithm of the allowance, and from the sum, subtract the logarithm of the given principal, the remainder is the logarithm of the rate per cent.
Ex. If the cost of Insuring L.1250, be 26L. 5s. what is the rate per cent.

$$2 + \text{Log. } 26.25 = 3.419129$$
$$\text{Log. } 1250 = \underline{3.096910}$$
$$\text{Log. } 2.1 = 0.322219$$
$$\text{L.}2.1 = 2 \text{ guineas. Ans.}$$

107. *To find the principal on which a given allowance is granted at a given rate per cent.* Add 2 to the logarithm of the given allowance, and from the sum subtract the logarithm of the given rate per cent., the remainder is the logarithm of the principal required.
Ex. What is the principal on which the commission is 49L. 17s. at 2½ per cent.

$$2 + \text{Log. } 46.85 = 3.670710$$
$$\text{Log. } 2.5 = \underline{0.397940}$$
$$\text{Log. } 1874 = 3.272770 \quad \text{Ans. L.}1874.$$

SIMPLE INTEREST.

108. *To find the interest, when the principal, rate, and time are given.* Add together the logarithms of the principal, rate, and time, and from the sum subtract 2, the remainder is the logarithm of the interest. *Note.* When the time is given in months, weeks, or days, the logarithm of the equivalent decimal of a year must be taken.
Ex. 1. What is the interest on L.875 for 3 years at 5 per cent.

$$\text{Log. } 875 = 2.942008$$
$$\text{Log. } 3 = 0.477121$$
$$\text{Log. } 5 = \underline{0.698970}$$
$$\text{Log. } 131.25 = 2.118099 \text{ sum}-2.$$
$$\text{L.}131.25 = 131\text{L. } 5\text{s. Interest. Ans.}$$

Ex. 2. What is the interest on L.570 for 219 days at 4 per cent.

$$\text{Log. } 570 = 2.755875$$
$$\text{Log. } .6 = \bar{1}.778151$$
$$\text{Log. } 4 = \underline{0.602060}$$
$$\text{Log. } 13.68 = 1.136086 \text{ sum}-2.$$
$$\text{L.}13.68 = 13\text{L. } 13\text{s. } 7\text{d. Ans.}$$

109. *To find the rate, when the interest, principal and time are given.* Add 2 to the logarithm of the interest, and from this sum, subtract the sum of the logarithms of the principal and time, the remainder is the logarithm of the rate.
Ex. At what rate per cent. will L.875 produce 135L. 5s. in 3 years.

$$\begin{aligned}\text{Log. } 875 &= 2.942008\\ \text{Log. } 3 &= \underline{0.477121}\\ & \quad 3.419129\\ 2 + \text{Log. } 131.25 &= \underline{4.118100}\\ \text{Log. } 5 &= 0.698971.\end{aligned}$$ Ans. 5 per cent.

110. *To find the time, when the interest, principal, and rate are given.* Add 2 to the logarithm of the interest, and from this sum, subtract the sum of the logarithms of the principal and rate, the remainder is the logarithm of the time.

Ex. In what time will L.570 produce 13L. 13s. $7\frac{1}{5}$d. at 4 per cent.

$$\begin{aligned}\text{Log. } 570 &= 2.755875\\ \text{Log. } 4 &= \underline{0.602060}\\ & \quad 3.357935\\ 2 + \text{Log. } 13.68 &= \underline{3.136086}\\ \text{Log. } .6 &= \bar{1}.778151\end{aligned}$$

.6 of a year = 219 days. Ans.

111. *To find the principal, when the interest, rate and time are given.* Add 2 to the logarithm of the interest, and from this sum, subtract the sum of the logarithms of the rate and time, the remainder is the logarithm of the principal.

Ex. What principal will produce L.126 in $3\frac{1}{2}$ years at 4 per cent.

$$\begin{aligned}\text{Log. } 4 &= 0.602060\\ \text{Log. } 3.5 &= \underline{0.544068}\\ & \quad 1.146128\\ 2 + \text{Log. } 126 &= \underline{4.100371}\\ \text{Log. } 900 &= 2.954243\end{aligned}$$ Ans. L.900.

CONTRACTION IN INTEREST.

112. *To find the interest for days at 5 per cent.* Add $\bar{4}.136677$ to the logarithms of the principal and the number of days, and the sum is the logarithm of the interest. *Note.* $\bar{4}.136677$ is the difference between the logarithm of 1, and the logarithm of 7300.

Ex. What is the interest of 295L. 12s. for 156 days.

$$\begin{aligned}\text{Log. } 1:7300 &= \bar{4}.136677\\ \text{Log. } 295.6 &= 2.470704\\ \text{Log. } 156 &= \underline{2.193125}\\ \text{Log. } 6.317 &= 0.800506\end{aligned}$$

L 6.317 = 6L. 6s. 4d. Ans.

113. The following is a short method of making the arithmetical calculation; "Multiply the principal by the number of days, and to the product add $\frac{1}{3}$ of itself, $\frac{1}{10}$ of this third, and $\frac{1}{10}$ of this tenth, and divide the sum by 10,000, the quotient is the interest required." From this

interest, one farthing for every 10L. 8s. 4½d. in it, must be deducted. The preceding example worked by this rule is as follows:

$$L.295.6 \times 156 = 46113.6$$
$$46113.6 \times \tfrac{1}{3} = 15371.2$$
$$15371.2 \times .1 = 1537.12$$
$$1537.12 \times .01 = \underline{153.712}$$
$$63175.632$$

63175 : 10000 = 6.3175 = 6L. 6s. 4d. Ans.

INVOLUTION.

114. *To find any power of a given number.* Multiply the logarithm of the given number by the index of the power, and the product is the logarithm of the power required. *Note.* When the index of the logarithm of the given number is negative, its product is negative; but if units be carried from the decimal part when multiplied, they are to be considered as positive.

Ex. 1. Required the square of 25.

$$\text{Log. } 25 = 1.397940$$
$$\underline{2 \text{ index}}$$
$$\text{Log. } 625 = 2.795880 \quad \text{Ans. } 625.$$

Ex. 1. Required the cube of .25.

$$\text{Log. } .25 = \bar{1}.397940$$
$$\underline{3 \text{ index}}$$
$$\text{Log. } .015625 = \bar{2}.193820 \quad \text{Ans. } .015625.$$

EVOLUTION.

115. *To find any root of a given number.* Divide the logarithm of the given number by the denominator of the index of the root, and the quotient is the logarithm of the root required. *Note.* When the index of the logarithm of the given number is negative, and not exactly divisible by the denominator of the index of the root, add as many units to the former as shall make it exactly divisible by the latter, but carry the same number of units to the decimal part of the logarithm, when divided.

Ex. 1. Required the square root of 4096.

$$2 \,|\, \underline{3.612360} = \text{Log. } 4096$$
$$\text{Log. } 64 = 1.806180 \quad \text{Ans. } 64.$$

Ex. 2. Required the cube root of .015625.

$$3 \,|\, \underline{\bar{2}.193820} = \text{Log. } .015625$$
$$\text{Log. } .25 = \bar{1}.397940 \quad \text{Ans. } .25.$$

Ex. 3. Required the fifth root of the fourth power of 1.05.

$$\text{Log. } 1.05 = 0.021189$$
$$\underline{4}$$
$$5 \,|\, \underline{0.084756}$$
$$\text{Log. } 1.0398 = 0.016951 \quad \text{Ans. } 1.0398.$$

COMPOUND INTEREST.

116. *To find the amount and interest, when the principal, rate and time are given.* Multiply the logarithm of the ratio by the time, to the product add the logarithm of the principal, and the sum is the logarithm of the amount. From the amount subtract the principal, and the remainder is the interest. *Note.* The ratio is the amount of L.1 for the first period, at the given rate per cent.

Ex. 1. What is the amount and interest of 216L. 10s. for 8 years at 5 per cent. interest payable yearly.

$$\begin{array}{rl}
\text{Log. } 1.05 = & 0.021189 \\
 & \underline{\qquad\quad 8} \text{ years.} \\
 & 0.169512 \\
\text{Log. } 216.5 = & \underline{2.335458} \\
\text{Log. } 319.869 = & 2.504970
\end{array}$$

L.319.869 = Amount 319L. 17s. 4½d. Interest 103L. 7s. 4½d.

Ex. 2. What is the interest of a penny, from the commencement of the Christian era till the present year (1833), at 5 per cent. interest payable yearly; and how many solid globes of pure gold, each as large as the earth, would that interest purchase, supposing that its mean diameter is 7912 miles, and the weight of a cubic foot of pure gold is 17591 ounces Troy, the gold being valued at L.3 : 17 : 10½ per ounce.

$$\begin{array}{rr}
\text{Log. } 1.05 = & 0.021189 \\
 & \underline{1833} \\
 & 63567 \\
 & 63567 \\
 & 169512 \\
 & \underline{21189} \\
 & 38.839437 \\
\text{Log. } 240 = & \underline{2.380211} \\
A = & 36.459226 \\
 & \underline{27.417419} \\
C = & 9.041807
\end{array}$$

$$\begin{array}{rr}
\text{Log. } 7912 = & 3.898286 \\
\text{Log. } .5280 = & \underline{3.722634} \\
 & 7.620920 \\
 & \underline{3} \\
 & 22.862760 \\
\text{Log. } .5236 = & \bar{1}.719000 \\
\text{Log. } 17591 = & 4.245291 \\
\text{Log. } 3.89375 = & \underline{0.590368} \\
B = & 27.417419
\end{array}$$

A is the logarithm of the amount or interest of 1d. for the given time; B is the logarithm of the value of a solid globe of gold, of the given diameter; and C is the logarithm of the number of such globes required. Thus, it appears that *the compound interest of a penny for* 1833 *years, would purchase upwards of One Thousand, one hundred and one Millions of solid globes of pure gold, each as large as the earth!*

117. *To find the principal and interest, when the amount, rate and time are given.* Multiply the logarithm of the ratio by the time, and subtract the product from the logarithm of the amount, the remainder is the logarithm of the principal; then, find the interest as in art. 116.

Ex. What principal would amount to 2164L. 11s. 6d. in 14 years at 4 per cent. interest payable yearly.

$$\text{Log. } 1.04 = 0.017033$$
$$\underline{\qquad 14 \text{ years}}$$
$$0.238462$$
$$\text{Log. } 2164.575 = \underline{3.335373}$$
$$\text{Log. } 1250 = \overline{3.096911} \quad \text{Ans. L. } 1250.$$

118. *To find the rate, when the principal, amount or interest, and time are given.* Subtract the logarithm of the principal from that of the amount, and divide the remainder by the time, the quotient is the logarithm of the ratio.

Ex. At what rate per cent. will L.500 amount to 814L. 8s. 10d. in 10 years, interest payable yearly.

$$\text{Log. } 814.441667 = 2.910860$$
$$\text{Log. } 500 = 2.698970$$
$$10\,|\,\overline{0.211890}$$
$$\text{Log. } 1.05 = \overline{0.021189} \quad \text{Ans. 5 per cent.}$$

119. *To find the time, when the principal, amount or interest, and rate are given.* Subtract the logarithm of the principal from that of the amount, and from the logarithm of the remainder subtract the logarithm of the logarithm of the ratio, the remainder is the logarithm of the time.

Ex. In what time will money double itself, at 5 per cent. per annum, interest payable yearly.

$$\text{Log. } 2 = 0.301030$$
$$\text{Log. } 1 = \underline{0.000000}$$
$$\text{Log. } 0.301030 = \bar{1}.478609$$
$$\text{Log. (Log. } 1.05) = \text{Log. } 0.021189 = \underline{\bar{2}.326111}$$
$$\text{Log. } 14.2069 = 1.152498$$
$$\text{Years } 14.2069 = 14 \text{ years } 75 \text{ days. Ans.}$$

ANNUITIES.

120. *To find the amount of an annuity at compound interest, forborn for a given time, at a given rate per cent.* Multiply the logarithm of the ratio by the time, and from the number indicated by the remainder, subtract unity; then, to the logarithm of this remainder add the logarithm of the given annuity, and from the sum subtract the logarithm of the difference between ratio and unity, the remainder is the logarithm of the amount required.

Ex. What is the amount of an annuity of L.20 payable yearly, forborn for 12 years, at 5 per cent.

$$\text{Log. } 1.05 = 0.021189$$
$$\underline{\qquad 12}$$
$$\text{Log. } 1.795843 = 0.254268$$
$$\underline{1.}$$
$$.795843$$

$$\text{Log. } .795843 = \bar{1}.900828$$
$$\text{Log. } 20 = \underline{\bar{1}.301030}$$
$$1.201858$$
$$\text{Log. } .05 = \underline{\bar{2}.698970}$$
$$\text{Log. } 318.3375 = 2.502888$$

L.318.3375 = 318L. 6s. 9d. Ans.

121. *To find the present value of an annuity at compound interest, for a given time, at a given rate per cent.* Find the amount of an annuity forborn for the given time, as above; then, from the logarithm of this amount subtract the product of the logarithm of the ratio by the time, and the remainder is the logarithm of the present value required.

Ex. What is the present value of an annuity of L.20 payable yearly, to last 12 years, at 5 per cent.

$$\left.\begin{array}{l}\text{Log. amount} = 2.502888 \\ \text{Log. product} = \underline{0.254268}\end{array}\right\}\text{ see last question.}$$

$$\text{Log. } 177.264 = 2.248620 \text{ remainder.}$$

$$\text{L.}177.264 = 177\text{L. } 5\text{s. } 3\text{d. Ans.}$$

122. *To find the present value of a perpetuity at compound interest, at a given rate per cent.* Subtract the logarithm of the difference between the ratio and unity, from the logarithm of the given annuity, and the remainder is the logarithm of the present value required.

Ex. What is the present value of a perpetual annuity of L.80 per annum, at 5 per cent.

$$\text{Log } 80 = 1.903090$$
$$\text{Log. } .05 = \underline{\bar{1}.698970}$$
$$\text{Log. } 1600 = 3.204120 \text{ Ans. L.1600.}$$

123. *To find the present value of annuities in reversion.* Rule I. *When perpetual.* From the logarithm of the present value as found by art. 122, subtract the product of the logarithm of the ratio by the time before it becomes payable, and the remainder is the logarithm of the present value of the perpetuity in reversion. II. *When limited.* Multiply the logarithm of the ratio by the time before, and the sum of the times before and after, the annuity becomes payable, and find the complements of the products to zero; then, to the logarithm of the difference of the numbers indicated by these complements, add the logarithm of the present value as found by art 122, the sum is the logarithm of the present value of the annuity in reversion.

Ex. 1. What is the present value of a perpetual annuity of L.80, which becomes payable, 10 years hence, at 5 per cent.

$$\text{Log. perpetuity} = 3.204120 \text{ see last question.}$$
$$\text{Log. } 1.05 \times 10 = \underline{0.211890}$$
$$\text{Log. } 982.268 = 2.992230 \text{ Ans. } 982\text{L. } 5\text{s. } 4\tfrac{1}{4}\text{d.}$$

Ex. 2. What is the present value of an annuity of L.20 per annum, to commence 10 years hence, and to continue 20 years afterwards, at 5 per cent.

$$\text{Log. } 1.05 \times 10 = 0.211890 \qquad \text{Log. } 1.05 \times 30 = 0.635670$$
$$\text{Co-Log. } .613917 = \bar{1}.788110 \qquad \text{Co-Log. } .231382 = \bar{1}.364330$$
$$\text{Log. } .382535 = \bar{1}.582671$$
$$\text{Log. perpetuity} = \underline{2.602060} \text{ by art. 122.}$$
$$\text{Log. } 153.014 = 2.184731 \text{ Ans. } 153\text{L. } 0\text{s. } 3\tfrac{1}{4}\text{d.}$$

MENSURATION BY LOGARITHMS.

LINES.

124. *To find one leg of a right-angled triangle, when the hypotenuse and the other leg are given.* Add together the logarithms of the sum and difference of the hypotenuse and the given leg, divide the sum by 2, and the remainder is the logarithm of the other leg.

Ex. Given the hypotenuse 505, and the base 303, to find the perpendicular.

$$\begin{array}{r} \text{Log. } 808 = 2.907411 \\ \text{Log. } 202 = \underline{2.305351} \\ 2 \,|\, \underline{5.212762} \\ \text{Log. } 404 = 2.606381 \end{array} \quad \text{Ans. } 404.$$

125. *To find the hypotenuse of a right-angled triangle when the two legs are given.* Add 10 to the logarithm of the perpendicular, from the sum subtract the logarithm of the base, the remainder is the logarithmic tangent of the angle at the base; then, from the same sum subtract the logarithmic sine of that angle, and the remainder is the logarithm of the hypotenuse.

Ex. Given the base 315, and the perpendicular 420, to find the hypotenuse.

$$\begin{array}{rr} 10 + \text{Log.}420 = 12.623249 & 12.623249 \\ \text{Log.}315 = \underline{\ 2.498311} & \text{Log. Sin. } 53^\circ\, 7'.8 = \underline{9.903189} \\ \text{Log.Tan. } 53^\circ\, 7'.8 = 10.124938 & \text{Log. } 525 = 2.720160 \end{array}$$

Hypotenuse 525. Ans.

126. *To find the diagonal of a square whose side is given.* Add 0.150515 to the logarithm of the given side, the sum is the logarithm of the diagonal.

Ex. Given the side of a square 234, to find its diagonal.

$$\begin{array}{r} \text{Log } 234 = 2.369216 \\ \tfrac{1}{2} \text{ Log. } 2 = \underline{0.150515} \\ \text{Log. } 330.926 = 2.519731 \end{array} \quad \text{Ans. } 330.926.$$

127. *To find the side of a square whose diagonal is given.* Add 0.150515 to the logarithm of half the given diagonal, the sum is the logarithm of the side.

Ex. Given the diagonal of a square 325, to find the side.

$$\begin{array}{r} \text{Log. } 162.5 = 2.210853 \\ \tfrac{1}{2} \text{ Log. } 2 = \underline{0.150515} \\ \text{Log. } 229.8095 = 2.361368 \end{array} \quad \text{Ans. } 229.8095.$$

128. *Given the diameter of a circle, to find its circumference.* Add 0.497150 to the logarithm of the diameter, the sum is the logarithm of the circumference:

Ex. Given the diameter 113, to find the circumference.

$$\begin{array}{r} \text{Log. } 113 = 2.053078 \\ \text{Log. } \pi = \underline{0.497150} \\ \text{Log. } 355 = 2.550228 \end{array} \quad \text{Ans. } 355.$$

129. *Given the circumference of a circle, to find its diameter.* Add $\bar{1}.502850$ to the logarithm of the circumference, the sum is the logarithm of the diameter.

Ex. Given the circumference 22, to find the diameter.

$$\begin{array}{ll} \text{Log.} & 22 = 1.342423 \\ \text{Log.} & 1:\pi = \underline{\bar{1}.502850} \\ \text{Log. } 7.00282 & = 0.845273 \end{array}$$ Ans. 7.00282.

130. *Given the diameter of a circle, to find the side of a square of equal area.* Add $\bar{1}.947545$ to the logarithm of the diameter, the sum is the logarithm of the side of the square required.

Ex. Given the diameter of a circle 20, to find the side of the equal square.

$$\begin{array}{l} \text{Log. } 20 = 1.301030 \\ \tfrac{1}{2}\text{ Log. } \tfrac{1}{4}\pi = \underline{\bar{1}.947545} \\ 17.7245 = 1.248575 \end{array}$$ Ans. 17.7245.

131. *Given the circumference of a circle, to find the side of a square of equal area.* Add $\bar{1}.450395$ to the logarithm of the circumference, the sum is the logarithm of the square required.

Ex. Given the circumference of a circle 355, to find the side of the equal square.

$$\begin{array}{l} \text{Log. } 355 = 2.550228 \\ \tfrac{1}{2}\text{ Log. } 1:4\pi = \underline{\bar{1}.450395} \\ \text{Log. } 100.144 = 2.000623 \end{array}$$ Ans. 100.144.

132. *Given the side of a square, to find the diameter of a circle of equal area.* Add 0.052455 to the logarithm of the side; the sum is the logarithm of the diameter required.

Ex. Given the side of a square 90, to find the diameter of the equal circle.

$$\begin{array}{l} \text{Log. } 90 = 1.954243 \\ \tfrac{1}{2}\text{ Log. } 4:\pi = \underline{0.052455} \\ \text{Log. } 101.554 = 2.006698 \end{array}$$ Ans. 101.554.

133. *Given the side of a square, to find the circumference of a circle of equal area.* Add 0.549605 to the logarithm of the side, the sum is the logarithm of the diameter required.

Ex. Given the side of a square 120, to find the circumference of the equal circle.

$$\begin{array}{l} \text{Log. } 120 = 2.079181 \\ \tfrac{1}{2}\text{ Log. } 4\pi = \underline{0.549605} \\ \text{Log. } 425.389 = 2.628786 \end{array}$$ Ans. 425.389.

Note. Several of the preceding problems may be easily solved by the inspection of Table VIII, particularly when the data are within the limits of the nine digits.

134. *To find the length of any arc of a circle to radius unity.* Add $\bar{2}.241877$ to the logarithm of the number of degrees in the arc, the sum is the logarithm of the length required.

Note. When the arc contains minutes, seconds, &c. of a degree, their decimal equivalent must be taken.

Ex. What is the length of the arc of 57° 17′ 44″ 48‴.

$$\begin{array}{ll}
\text{Log. } 57°.2957 & = 1.758123 \\
\text{Log. } \pi{:}180° & = \bar{1}.241877 \\
\hline
\text{Log. } 1 & = 0.000000 \quad \text{Ans. } 1.
\end{array}$$

Otherwise, by Table IX.

Length of 50° =	.872665
………… 7° =	.122173
………… 10′ =	2909
………… 7′ =	2036
………… 40″ =	194
………… 4″ =	19
………… 48‴ =	4
Length of 57° 7′ 44″ 48‴ =	1.000000

Note. The preceding rule is only applicable, ***when the radius is unity.*** In all other cases, the logarithm of the length of the arc, is equal to the sum of the logarithms of its radius, and its length found as above.

135. *Given the length of an arc of a circle, to find the number of degrees it contains.* Add 1.758123 to the logarithm of the length of the arc, and from the sum subtract the logarithm of its radius, the remainder is the logarithm of the number of degrees required.

Ex. How many degrees are contained in the arc whose length is 2, and radius 1.

$$\begin{array}{ll}
\text{Log. } 2 & = 0.301030 \\
\text{Log. } 180°{:}\pi & = 1.758123 \\
\hline
\text{Log. } 114.592 & = 2.059153 \quad \text{Ans. } 114° \ 35' \ 30''.
\end{array}$$

Otherwise, by Table IX.

	2.000000
Length of 110° =	1.919862
	.080138
………… 4° =	.069813
	.010325
………… 30′ =	8727
	.001598
………… 5′ =	1454
	.000144
………… 30″ =	145

Ans. 114° 35′ 30″ nearly.

136. ***Given the chord and sagitta of any arc or segment of a circle to find the diameter.*** Multiply the logarithm of half the chord by 2, and from the product subtract the logarithm of the sagitta, the remainder is the logarithm of the sagitta of the explementary arc; the sum of the two sagittas is the diameter required. *Note.* The *sagitta* of an arc or segment is the distance from the middle of the chord to the middle of the arc.

Ex. Given the chord of an arc 16, and its sagitta 4, to find the diameter.

$$2 \times \log. 8 = 1.806180$$
$$\text{Log. } 4 = 0.602060$$
$$\text{Log. } 16 = 1.204120 \quad \text{Ans. } 20.$$

Note. When the number of degrees in a given arc is a multiple of 5, its chord and sagitta will be found by the inspection of Table X.

137. *Given the altitude of the centre of the visible horizon in feet, to find its semidiameter in miles.* To the logarithm of the given altitude add 0.176091, and divide the sum by 2, the quotient is the logarithm of the semidiameter required.

Ex. How far could a spectator see objects at the level of the sea, independently of refraction, from the summit of Benlomond, its height above that level being estimated at 3262 feet.

$$\text{Log. } 3262 = 3.513484$$
$$\text{Log. } 1.5 = 0.176091$$
$$2 \,|\, 3.689575$$
$$\text{Log. } 69.95 = 1.844788 \quad \text{Ans. } 69.95 \text{ miles.}$$

138. This rule, which is only a very near approximation, but sufficiently accurate for practice, may be otherwise expressed thus: *The semidiameters of the visible horizon in leagues are as the square roots of the altitudes of its centre in fathoms;* that is, supposing the altitudes to be 1, 4, 9, 16, 25, &c. fathoms, the semidiameters will be 1, 2, 3, 4, 5, &c. leagues.

139. *Given the altitude of the centre of the visible horizon in feet, to find at what distance in miles, all objects of a certain altitude (given in feet) will be visible.* Find the semidiameter of the visible horizon for both altitudes, as above, and the sum of these semidiameters is the distance required.

Ex. From the top of a mast of a vessel, the peak of Teneriffe was just seen peering above the horizon; at what distance was the vessel from the mountain, supposing the top of the mast 100 feet, and the summit of Teneriffe 12000 feet above the level of the sea.

Log. 100 = 2.000000	Log. 12000 = 4.079181
Log. 1.5 = 0.176091	Log. 1.5 = 0.176091
2 | 2.176091	2 | 4.255272
Log. 12.25 = 1.088046	Log. 134.16 = 2.127636

$$12.25 + 134.16 = 146.41 \text{ miles. Ans.}$$

140. *Given the semidiameter of the visible horizon in miles, to find the altitude of its centre in feet.* Multiply the logarithm of the given semidiameter by 2, and to the product add $\bar{1}.823909$, the sum is the logarithm of the altitude required.

Ex. If the summit of Mont Blanc be just visible from the level of the sea at Nice, what is its altitude, supposing it to be 153 miles distant from the town.

$$\begin{aligned} \text{Log. } 153 &= 2.184691 \\ & \quad\quad\quad\; 2 \\ \hline & \, 4.369382 \\ \text{Log. } 2{:}3 &= \bar{1}.823909 \\ \hline \text{Log. } 15606 &= 4.193291 \quad \text{Ans. } 15606 \text{ feet.} \end{aligned}$$

141. Conversely to art. 138, *The altitudes of the centre of the visible horizon in fathoms, are as the squares of the semidiameters in leagues;* that is, supposing the semidiameters to be 1, 2, 3, 4, 5, &c. leagues, the altitudes will be 1, 4, 9, 16, 25, &c. fathoms.

SURFACES.

142. *To find the area of a parallelogram.* The logarithm of the area is equal, *First*, to the sum of the logarithms of the base and altitude; *Secondly*, to the sum of the logarithms of any two adjacent sides, and the logarithmic sine of their included angle, minus 10; or, *Thirdly*, to the sum of the logarithms of the two diagonals, the logarithmic sine of their intersected angle, and $\bar{1}\bar{1}.698970$.

Ex. 1. Required the area of a parallelogram whose base is 316 and altitude 125.

$$\begin{aligned} \text{Log. } 316 &= 2.499687 \\ \text{Log. } 125 &= 2.096910 \\ \hline \text{Log. } 39500 &= 4.596597 \quad \text{Ans. } 39500. \end{aligned}$$

Ex. 2. Required the area of a parallelogram whose adjacent sides are 316 and 125, and included angle 42° 50′.

$$\begin{aligned} \text{Log. } 316 &= 2.499687 \\ \text{Log. } 125 &= 2.096910 \\ \text{Log. Sin. } 42^\circ\, 50' &= 9.832425 \\ \hline \text{Log. } 26854.8 &= 4.429022 \quad \text{Ans. } 26854.8. \end{aligned}$$

Ex. 3. Required the area of a parallelogram whose diagonals are 216 and 105, and angle of intersection 53° 10′.

$$\begin{aligned} \text{Log. } 216 &= \;\; 2.335458 \\ \text{Log. } 105 &= \;\; 2.021189 \\ \text{Log Sin. } 53^\circ\, 10' &= \;\; 9.903298 \\ (\text{Log. } \tfrac{1}{2}) - 10 &= \bar{1}\bar{1}.698970 \\ \hline \text{Log. } 9097.35 &= \;\; 3.958915 \quad \text{Ans. } 9097.35. \end{aligned}$$

143. *To find the area of a triangle.* The logarithm of the area of a triangle is equal, *First*, to the sum of the logarithms of the base and altitude, and $\bar{1}.698970$; *Secondly*, to the sum of the logarithms of any two adjacent sides, the logarithmic sine of their included angle, and $\bar{1}\bar{1}.698970$; or, *Thirdly*, to half the sum of the logarithms of the semiperimeter and the differences between each side and the semiperimeter.

Ex. 1. Required the area of a triangle whose base is 320, and altitude 280.

$$\begin{aligned} \text{Log. } 320 &= 2.505150 \\ \text{Log. } 280 &= 2.447158 \\ \text{Log. } .5 &= \bar{1}.698970 \\ \hline \text{Log. } 44800 &= 4.651278 \quad \text{Ans. } 44800. \end{aligned}$$

Ex. 2. Required the area of a triangle whose adjacent sides are 300 and 200, and their included angle 70° 36′.

$$\begin{aligned} \text{Log. } 300 &= 2.477121 \\ \text{Log. } 200 &= 2.301030 \\ \text{Log. Sin. } 70° 36' &= 9.974614 \\ (\text{Log. } \tfrac{1}{2}) - 10 &= \bar{1}\bar{1}.698970 \\ \hline \text{Log. } 28296.6 &= 4.451735 \quad \text{Ans. } 28296.6. \end{aligned}$$

Ex. 3. Required the area of a triangle whose three sides are 75, 85, and 95.

$$\begin{aligned} \text{Log. } 127.5 &= 2.105510 \\ \text{Log. } 52.5 &= 1.720159 \\ \text{Log. } 42.5 &= 1.628389 \\ \text{Log. } 32.5 &= 1.511883 \\ \hline 2 &\,|\, 6.965941 \\ \hline \text{Log. } 3040.68 &= 3.482971 \quad \text{Ans. } 3040.68. \end{aligned}$$

144. *To find the area of any polygon.* Divide the polygon into triangles, by drawing straight lines, *either* from one of its angles to all the rest, except the two adjacent angles: *or*, from any assumed point within the figure to all its angles; then, calculate the areas of these triangles by any of the preceding methods, and their sum will be the area of the polygon.

Ex. Required the area of a polygon, in which a point O was assumed, and the following straight lines drawn to all its angles: OA = 120, OB = 150, OC = 100, OD = 170, and OE = 200; the angles formed by these straight lines at the point O, being as follows: AOB = 70° 30′, BOC = 60° 20′, COD = 50° 40′, DOE = 110° 10′, EOA = 68° 20′. Ans. 48685.

145. *To find the area of any regular polygon, of which a side is given.* Double the logarithm of the given side, and to the product add the logarithm of the area of a similar polygon whose side is unity, taken from Table XIV., the sum is the logarithm of the area required.

Ex. Required the area of a regular pentagon whose side is 12.

$$\begin{aligned} 2 \times \text{Log. } 12 &= 2.158362 \\ \text{Tabular Log.} &= 0.235649 \\ \hline \text{Log. } 247.749 &= 2.394011 \quad \text{Ans. } 247.749. \end{aligned}$$

146. *To find the area of a circle.* The logarithm of the area is equal, *First*, to the sum of the logarithms of the semidiameter and the semicircumference; *Secondly*, to the

sum of double the logarithm of the diameter, and $\bar{1}.895090$; *Thirdly*, to the sum of double the logarithm of the circumference, and $\bar{2}.900790$.

Ex. 1. Required the area of a circle whose diameter is 113, and circumference 355.

$$\begin{array}{r} \text{Log. } 56.5 = 1.752048 \\ \text{Log. } 177.5 = \underline{2.249198} \\ \text{Log. } 10028.73 = 4.001246 \end{array} \quad \text{Ans. } 10028.73.$$

Ex. 2. Required the area of a circle whose diameter is 113.

$$\begin{array}{r} 2 \times \text{Log. } 113 = 4.106156 \\ \text{Log. } \tfrac{1}{4}\pi = \underline{\bar{1}.895090} \\ \text{Log. } 10028.73 = 4.001246 \end{array} \quad \text{Ans. } 10028.73.$$

Ex. 3. Required the area of a circle whose circumference is 355.

$$\begin{array}{r} 2 \times \text{Log. } 355 = 5.100456 \\ \text{Log. } 1 : 4\pi = \underline{\bar{2}.900790} \\ \text{Log. } 10028.73 = 4.001246 \end{array} \quad \text{Ans. } 10028.73.$$

147. *Given the area of a circle, to find its diameter.* To the logarithm of the area, add 0.104910, and divide the sum by 2, the quotient is the logarithm of the diameter.

Ex. Required the diameter of a circle whose area is half an acre.

$$\begin{array}{r} \text{Log. } 2420 = 3.383815 \\ \text{Log. } 4 : \pi = \underline{0.104910} \\ 2 \,|\, 3.488725 \\ \text{Log. } 55.509 = 1.744363 \end{array} \quad \text{Ans. } 55.509 \text{ yds.}$$

148. *Given the area of a circle, to find its circumference.* To the logarithm of the area, add 1.099210, and divide the sum by 2, the quotient is the logarithm of the circumference.

Ex. Required the circumference of a circle, whose area is an acre.

$$\begin{array}{r} \text{Log. } 4840 = 3.684845 \\ \text{Log. } 4\pi = \underline{1.099210} \\ 2 \,|\, 4.784055 \\ \text{Log. } 246.62 = 2.392028 \end{array} \quad \text{Ans. } 246.62 \text{ yds.}$$

149. *To find the area of a circular ring.* To the logarithms of the sum and difference of the inner and outer diameters, add $\bar{1}.895090$, and the sum is the logarithm of the area required.

Ex. Required the area of a ring whose inner diameter is 10, and outer diameter 12.

$$\begin{array}{r} \text{Log. } 22 = 1.342423 \\ \text{Log. } 2 = 0.301030 \\ \text{Log. } \tfrac{1}{4}\pi = \underline{\bar{1}.895090} \\ \text{Log. } 34.5575 = 1.538543 \end{array} \quad \text{Ans. } 34.5575.$$

150. *To find the area of a sector and segment of a circle.* The sum of the logarithms of the radius and half the length of the arc, is the logarithm of the *area of the sector;* and the sum or difference of this area, and that of the triangle whose base is the chord of the arc, and sides the radii of the sector, is the *area of the segment,* according as it is greater or less than a semicircle.

Ex. Required the area of the sector and segment of a circle, whose arc is equal to radius, and radius 10.

$$\begin{array}{lrl} & \text{Log. } 5 = & 0.698970 \\ & \text{Log. } 10 = & \underline{1.000000} \\ & \text{Log. } 50 = & 1.698970 \\ \text{Hence,} & 50.00000 = & \text{area of the sector.} \\ \text{And,} & \underline{43.30127} = & \text{area of the triangle.} \\ \text{Therefore,} & 6.69873 = & \text{area of the segment.} \end{array}$$

Note. The area of a circular segment may be found by Table XVI. as follows: Divide the sagitta by the diameter of the circle, and look in the column marked S. for the quotient; multiply the number in the column marked *area*, which stands on the right of this quotient, by the square of the diameter, and the product will be the area required.

Ex. What is the area of a circular segment whose sagitta is 10, and radius 25.

$$\begin{array}{lrl} & .111823 = & \text{area to sagitta } .2 \\ 10 \div 50 = .2 & \underline{2500} & \\ & 279.557500 = & \text{area required.} \end{array}$$

151. *To find the area of an ellipse.* To the logarithms of the major and minor axes, add $\bar{1}.895090$, and the sum is the logarithm of the area required.

Ex. Required the area of an ellipse whose major axis is 56, and minor axis 42.

$$\begin{array}{rl} \text{Log. } 56 = & 1.748188 \\ \text{Log. } 42 = & 1.623249 \\ \text{Log. } \frac{1}{4}\pi = & \underline{\bar{1}.895090} \\ \text{Log } 1847.25 = & 3.266527 \quad \text{Ans. } 1847\tfrac{1}{4}. \end{array}$$

152. *To find the area of a parabola.* To the logarithms of the given abscissa and its double ordinate, add $\bar{1}.823909$, and the sum is the logarithm of the area required.

Ex. Required the area of a parabola whose abscissa is 6, and double ordinate 19.

$$\begin{array}{rl} \text{Log. } 6 = & 0.778151 \\ \text{Log. } 19 = & 1.278754 \\ \text{Log. } 2{:}3 = & \underline{\bar{1}.823909} \\ \text{Log. } 76 = & 1.880814 \quad \text{Ans. } 76. \end{array}$$

153. *To find the surface of a sphere.* To double the logarithm of the diameter, add 0.497150, and the sum will be the logarithm of the surface required. *Note.* The loga-

rithm of the convex surface of any spherical zone or segment is equal to the sum of the logarithms of its height and the diameter, and 0.497150.

Ex. 1. Required the number of square miles in the surface of the earth, supposing its mean diameter to be 7912 miles.

$$2 \times \text{Log. } 7912 = 7.796572$$
$$\text{Log. } \pi = 0.497150$$
$$\text{Log. } 196662730 = 8.293722$$

Ans. 196662730 square miles nearly.

Ex. 2. Required the number of square miles on the surface of the earth, visible from the summit of Chimborazo, supposing its altitude to be 21000 feet above the level of the sea.

Log. 21000 = 4.322219	Log. Cos. 2° 34′ = 9.999564
Log. 1.5 = 0.176091	10.000000
2 | 4.498310	L. N. Cos. .999 = $\bar{1}$.999564
2.249155	L. N. V. Sin. .001 = $\bar{3}$.000000
Log. 3956 = 3.597256	Log. 3956 = 3.597256
$\bar{2}$.651899	Log. 7912 = 3.898286
Log. Rad. = 10.000000	Log. π = 0.497150
Log. Tan. 2° 34′ = 8.651899	Log. 98331 = 4.992692

Ans. 98331 square miles nearly.

SOLIDS.

154. *To find the capacity of a parallelepiped.* The sum of the logarithms of the length, breadth, and depth (or thickness), is the logarithm of the capacity.

Ex. Required the capacity of a cistern whose length is 16 feet, breadth 5¾ feet, and depth 3½ feet.

$$\text{Log } 16 = 1.204120$$
$$\text{Log. } 5.75 = 0.759668$$
$$\text{Log. } 3.5 = 0.544068$$
$$\text{Log. } 322 = 2.507856$$

Ans. 322 cub. feet.

155. *Given the capacity and two dimensions of a parallelepiped, to find the third.* From the logarithm of the capacity, subtract the sum of the logarithms of the given dimensions, the remainder is the logarithm of the third.

Ex. What must be the length of a reservoir to hold 400 Imperial gallons of oil, whose breadth is 3 feet, and depth 2½ feet.

Log. 400 = 2.602060	Log. 3 = 0.477121
Log. 277.274 = 2.442909	Log. 2.5 = 0.397940
5.044969	0.875061
Log. 1728 = 3.237544	Log. capacity = 1.807425
Log. capacity = 1.807425	Log. 8.5578 = 0.932364

Ans. 8.5578 feet.

156. *Given the capacity of a cube, to find its side.* Divide the logarithm of the capacity by 3, the quotient is the logarithm of the side.

Ex. What is the side of a cubical vessel to hold 10000 imperial gallons of water.

$$\begin{array}{rl}
\text{Log. } 10000 = & 4.000000 \\
\text{Log. } 277.274 = & \underline{2.442909} \\
 & 6.442909 \\
\text{Log. } 1728 = & \underline{3.237544} \\
3\,| & 3.205365 \\
\text{Log. } 11.707 = & 1.068455 \quad \text{Ans. } 11.707 \text{ feet.}
\end{array}$$

157. *To find the capacity of a prism or cylinder.* To the logarithm of the area of the base, add the logarithm of the altitude (or depth), the sum is the logarithm of the capacity.

Ex. 1. Required the capacity of a triangular prism, the edges of the base being $3\frac{1}{2}$, $4\frac{1}{2}$, and $5\frac{1}{2}$ inches respectively, and the length 1 foot and a half.

$$\begin{array}{rl}
\text{Log. area of base} = & 0.896182 \\
\text{Log. } 18 \text{ inches} = & \underline{1.255273} \\
\text{Log. } 141.727 = & 2.151455
\end{array}$$

Ans. 141.727 cubic inches.

Ex. 2. Required the capacity of a cylinder, the diameter of the base being 21 inches, and the length $4\frac{1}{2}$ feet.

$$\begin{array}{rl}
\text{Log. area of base} = & 2.539528 \\
\text{Log. } 54 \text{ inches} = & \underline{1.732394} \\
 & 4.271922 \\
\text{Log. } 1728 = & \underline{3.237544} \\
\text{Log. } 10.8237 = & 1.034378
\end{array}$$

Ans. 10.8237 cubic feet.

158. *Given the capacity of a cylinder, and its depth, to find the diameter of the base.* To the logarithm of the capacity, add 0.104910; from the sum subtract the logarithm of the depth, and half the remainder is the logarithm of the diameter required.

Ex. Required the interior diameter of the base of an imperial bushel, whose interior depth is 8 inches.

$$\begin{array}{rl}
\text{Log. } 2218.192 = & 3.346000 \\
\text{Log. } 4{:}\pi = & \underline{0.104910} \\
 & 3.450910 \\
\text{Log. } 8 = & \underline{0.903090} \\
2\,| & 2.547820 \\
\text{Log. } 18.7892 = & 1.273910 \quad \text{Ans. } 18.7892 \text{ in.}
\end{array}$$

159. *Given the capacity of a cylinder, and the diameter of its base, to find its depth.* To the logarithm of the capacity, add 0.104910; from the sum, subtract twice the logarithm of the diameter, and the remainder is the logarithm of the depth.

Ex. Required the interior depth of an imperial bushel, the interior diameter of the base being $18\frac{1}{2}$ inches.

$$\text{Log. } 2218.192 = 3.346000$$
$$\text{Log. } 4 : \pi = 0.104910$$
$$\overline{3.450910}$$
$$2 \times \text{Log. } 18.5 = 2.534344$$
$$\text{Log. } 8.25213 = \overline{0.916566} \text{ Ans. } 8.25213 \text{ in.}$$

160. *To find the capacity of a pyramid or cone.* To the logarithms of the area of the base, and the altitude, add $\bar{1}.522879$, and the sum is the capacity required.

Ex. 1. Required the weight of the largest pyramid of Egypt, which, according to Mr. Herschel, is composed of granite, is 700 feet in the side of its base, and 500 feet in perpendicular height, and stands on 11 acres of ground; supposing that a cubic foot of granite weighs 2654 ounces avoirdupois; that the pyramid is solid within, that its base is square, and that its faces are triangular planes.

Log. area of base $= 5.690196$	Log. capacity $= 7.912045$
Log. $500 = 2.698970$	Log. $2654 = 3.423901$
Log. $1 : 3 = \bar{1}.522879$	Co-Log. $35840 = 5.445632$
Log. capacity $= 7.912045$	Log. $6047528 = 6.781578$

Ans. 6047528 Tons.

Ex. 2. Required the capacity of a cone, the diameter of the base being 7, and the altitude 12.

$$\text{Log. area of base} = 1.585286$$
$$\text{Log. } 12 = 1.079181$$
$$\text{Log. } 1 : 3 = \bar{1}.522879$$
$$\text{Log. } 153.938 = \overline{2.187346} \text{ Ans. } 153.938.$$

161. *To find the capacity of a frustum of a pyramid or cone.* Divide the sum of the logarithms of the areas of the opposite bases by 2; to the number indicated by the quotient, add the areas of the bases; to the logarithm of their sum, and that of the altitude, add $\bar{1}.522879$, and the sum is the logarithm of the capacity required.

Ex. 1. Required the capacity of a frustum of a trigonal pyramid, the edges of the greater and less ends being 3 and 5 respectively, and the altitude 9.

Log. $9 = 0.954243$	Log. $3.8971 = 0.590744$
Tab. Log. $= \bar{1}.636501$	Log. $10.8253 = 1.034441$
Log. $3.8971 = 0.590744$	$2 \mid 1.625185$
Log. $25 = 1.397940$	Log. $6.4952 = 0.812593$
Tab. Log. $= \bar{1}.636501$	Log. $21.2176 = 1.326697$
Log. $10.8253 = 1.034441$	Log. $9 = 0.954243$
	Log. $1 : 3 = \bar{1}.522879$
Ans. 63.653 cubic feet.	Log. $63.653 = 1.803819$

Ex. 2. Required the capacity of the frustum of a cone,

whose altitude is 9, and whose top and bottom diameters are 3 and 5 respectively.

Log. 9 = 0.954243
Log. $\frac{1}{4}\pi$ = $\bar{1}$.895090
Log. 7.0686 = 0.849333
Log. 25 = 1.397940
Log. $\frac{1}{4}\pi$ = $\bar{1}$.895090
Log. 19.6349 = 1.293030

Ans. 114.454

Log. 7.0686 = 0.849333
Log. 19.6349 = 1.293030
2 | 2.142363
Log. 11.7810 = 1.071182
Log. 38.4845 = 1.585286
Log. 9 = 0.954243
Log. 1 : 3 = $\bar{1}$.522879
Log. 115.454 = 2.062408

162. *Given the capacity of the frustum of a cone, and its top and bottom diameters, to find its depth.* Divide the sum of the logarithms of the areas at top and bottom by 2; to the number indicated by the quotient, add the areas of the top and bottom; then, subtract the logarithm of this sum, from the logarithm of the capacity increased by 0.477121, the remainder is the logarithm of the depth.

Ex. Required the interior depth of an imperial bushel, whose interior diameters at top and bottom, are 18$\frac{3}{4}$ and 12 inches respectively.

Log. area top = 2.441092
Log. area bottom = 2.053452
2 | 4.494544
Log. 176.714 = 2.247272
276.117 = top area
113.097 = bot. area
565.928 = sum

Log. 2218.192 = 3.346000
Log. 3 = 0.477121
3.823121
Log. 565.928 = 2.752762
Log. 11.7587 = 1.070359

Ans. 11.7587 inches.

163. *To find the capacity of a sphere.* To 3 times the logarithm of the diameter, add $\bar{1}$.719000, and the sum is the logarithm of the capacity. *Note.* The logarithm of the capacity of a segment of a sphere, is equal to the sum of the logarithm of the difference between the radius of the sphere and $\frac{1}{3}$ of the height of the segment, twice the logarithm of this height, and 0.497150.

Ex. 1. What is the capacity of a sphere 5 inches in diameter.

3 × Log. 5 = 2.096910
Log. π : 6 = $\bar{1}$.719000
Log. 65.45 = 1.815910 Ans. 65.45 cub.in.

Ex. 2. What is the capacity of a segment of a sphere 1 foot in diameter, the height of the segment being 3 inches.

Log. 5 = 0.698970
2 × Log. 3 = 0.954243
Log. π = 0.497150
Log. 141.34 = 2.150363
Ans. 141.34 cubic inches.

164. *Given the capacity of a sphere, to find its diameter.*

To the logarithm of the capacity, add 0.281000, divide the sum by 3, and the quotient is the logarithm of the diameter.

Ex. What is the diameter of a sphere whose capacity is a cubic foot.

$$\begin{aligned} \text{Log. } 1728 &= 3.237544 \\ \text{Log. } 6 : \pi &= \underline{0.281000} \\ 3 \,|\, &\underline{3.518544} \\ \text{Log. } 14.8884 &= 1.172848 \text{ Ans. } 14.8884 \text{ in.} \end{aligned}$$

165. *To find the capacity of a spheroid.* To twice the logarithm of the revolving axis, add the logarithm of the fixed axis, and $\bar{1}.719000$; the sum is the logarithm of the capacity.

Ex. Required the difference of the capacities of the earth on the hypotheses of Cassini and Newton, the axes being 7925 miles and 7899 miles respectively.

2 Log. 7925= 7.797998	2 Log. 7899= 7.795144
Log. 7899= 3.897572	Log. 7925= 3.898999
Log. π : 6= $\bar{1}.719000$	Log. π : 6= $\bar{1}.719000$
Log.259758700000=11.414570	Log.258906500000=11.413143

cubic miles 259758700000 = capacity as a prolate spheroid.
cubic miles 258906500000 = capacity as an oblate spheroid.

cubic miles 852200000 = difference of capacities. Ans.

Thus Cassini's hypothesis makes the earth's capacity about 852 millions of cubic miles larger than Newton's.

166 *To find the capacity of the middle frustum of a parabolic spindle.* To twice the logarithm of the middle diameter, add 0.903090; to twice the logarithm of the end diameter, add 0.477121; to the sum of the logarithms of both diameters, add 0.602060; to the logarithm of the sum of the numbers indicated by these three results, add the logarithm of the length, and $\bar{1}.719000$; the sum is the logarithm of the capacity.

Ex. Required the number of imperial gallons in a cask of the form of a parabolic spindle, whose interior depth is 4 feet, and interior diameters at the end and in the middle, 2 feet and 3 feet respectively.

2 Log. 36 = 3.112606	10368
Log. 8 = 0.903090	1728
Log. 10368 = 4.015696	3456
2 Log. 24 = 2.760422	Log. 15552 = 4.191786
Log. 3 = 0.477121	Log. 48 = 1.681241
Log. 1728 = 3.237543	Log. π : 60 = $\bar{2}.719000$
Log. 36 = 1.556303	Log. capacity = 4.592027
Log. 24 = 1.380211	Log. 277.274 = 2.442909
Log. 4 = 0.602060	Log. 140.967 2.149118
Log. 3456 = 3.538574	Ans. 140.967 gallons.

167. *To find the capacity of any regular polyhedron.* To 3 times the logarithm of the length of the edge, add the logarithm of the capacity of a similar polyhedron, taken from Table XV., the sum is the logarithm of the capacity.
Ex. Required the capacity of an octahedron, whose linear edge is 12 inches.

$$\begin{aligned} 3 \text{ Log. } 12 &= 3\ 237544 \\ \text{Tab Log.} &= \bar{1}.673394 \\ \hline \text{Log. } 814\ 589 &= 2\ 910938 \end{aligned}$$

Ans. 814 589 cubic inches.

168. *To find the weight of a body, its capacity in cubic inches, and specific gravity being given.* To the logarithms of the capacity and specific gravity, add $\bar{5}$.557091, the sum is the logarithm of the weight in pounds avoirdupois.
Ex. Required the weight of a cast-iron beam 12 feet long, 8 inches deep, and 4 inches broad, the specific gravity of cast-iron being reckoned at 7425.

$$\begin{aligned} \text{Log. } 4\cdot 08 &= 3\ 663512 \\ \text{Log. } 7425 &= 3\ 870696 \\ \text{Log. } 1{:}277274 &= \bar{5}\ 557091 \\ \hline \text{Log. } 1234 &= 3\ 091299 \end{aligned}$$

Ans. 1234 lbs.

169. *To find the capacity of a body, its weight in pounds, and specific gravity being given.* To the logarithm of the weight, add 4.442909, and from the sum, subtract the logarithm of the specific gravity, the remainder is the logarithm of the capacity in cubic inches
Ex. Required the number of imperial gallons in a butt of Madeira, weighing nett 9 cwt. 1 qr. 5 lb., supposing the specific gravity of the wine to be 1038.

$$\begin{aligned} \text{Log. } 1041 &= 3.017451 \\ \text{Log. } 277274 &= 4.442909 \\ \hline &\ 7.460360 \\ \text{Log. } 1038 &= 3.016197 \\ \hline \text{Log. capacity} &= 4.444163 \\ \text{Log. } 277.274 &= 2.442909 \\ \hline \text{Log. } 100\ 3 &= 2\ 001254 \end{aligned}$$

Ans. 100.3 galls.

170. *To find the distance of a body which emits light and sound at the same instant* To the logarithm of the time in seconds, which elapses between seeing the light and hearing the sound, add $\bar{1}$.326584, the sum is the logarithm of the distance in miles.
Ex. Ten seconds after the appearance of a flash of lightning, the roll of thunder was heard; at what distance was the cloud from which they issued.

$$\begin{aligned} \text{Log. } 10 &= 1.000000 \\ \text{Log. } 1120 : 5280 &= \bar{1}\ 326584 \\ \hline \text{Log. } 2.121 &= 0.326584 \end{aligned}$$

miles 2.121 = 2 miles, 213 yards. Ans.

TRIGONOMETRY.

PLANE TRIANGLES.

171. Case I. *Given two sides of a triangle, and an angle opposite to one of the sides; or, two angles and a side; to find the remaining parts of the triangle.*

Rule 1. *To find an angle.* To the logarithmic sine of the given angle, add the arithmetical complement of the logarithm of the side opposite, and the logarithm of the side adjacent, to that angle; the sum diminished by 10, is the logarithmic sine of the angle opposite to the latter.

Rule 2. *To find a side.* To the logarithm of the given side, add the arithmetical complement of the logarithmic sine (or, the cosecant minus 10) of the angle opposite, and the logarithmic sine of the angle adjacent to that side; the sum minus 10, is the logarithm of the side opposite to the latter.

Note The angles of a triangle are generally denoted by the capital letters A, B, C; and the sides, by the small letters a, b, c.

Ex. In a triangle ABC, given $a = 104$ yards, $b = 120$ yards, and $B = 67° 23'$; to find the remaining parts of the triangle.

Log. sin. $67° 23' = 9.965248$	Log. $104 = 2.017033$
Co-Log. $120 = 7.920819$	Co-Log. sin. $53° 8' = 0.096900$
Log. $104 = \underline{2.017033}$	Log. sin. $59° 29' = \underline{9.935246}$
Log. sin. $53° 8' = 9.903100$	Log. $112 = 2.049179$

or, $126° 52'$. $A = 53° 8'$, $C = 59° 29'$, $c = 112$. Ans.

172. Case II. *Given two sides, and the included angle; to find the remaining parts of a triangle.*

Rule 1. *To find the other angles.* To the logarithmic tangent of half the sum of the angles opposite to the given sides, add the arithmetical complement of the logarithm of the sum of the sides, and the logarithm of their difference; the sum diminished by 10, is the logarithmic tangent of half the difference of those angles.

Rule 2. *To find the third side.* To the logarithmic cosine of half the sum of the angles opposite to the given sides, add the arithmetical complement of the logarithmic cosine of half their difference, and the logarithm of the sum of the sides; the sum diminished by 10, is the logarithm of the third side.

Note. The sum of the angles opposite to the given sides is found by subtracting the given angle from 180°; and the greater angle is found by adding half the difference to half the sum; the less, by subtracting the former quantity from the latter.

Ex. In a triangle, given $a = 290$, $b = 410$, and $C = 105°$; to find the rest.

Log. Tan. 37°30′ = 9.884980	Log. Cos. 37° 30′ = 9.899467
Co-Log. 700 = 7.154902	Co-Log. Cos. 7° 30′ = 0.003731
Log. 120 = 2.079181	Log. 700 = 2.845098
Log. Tan. 7°30′ = 9.119063	Log. 560.14 = 2.748296

A = 30° B = 44°, and c = 560.14. Ans.

173. Case III. *Given the three sides, to find the three angles of a triangle.*

Rule 1. To the logarithms of the differences between any two sides and the semiperimeter, add the arithmetical complements of the logarithms of these sides, and divide the sum by 2, the quotient is the logarithmic sine of half the angle opposite to the remaining side.

Rule 2. To the logarithms of the semiperimeter and the difference between it and any side, add the arithmetical complements of the logarithms of the other two sides, and divide the sum by 2, the quotient is the logarithmic cosine of half the angle opposite the former side.

Rule 3. To the logarithms of the differences between any two sides and the semiperimeter, add the arithmetical complements of the logarithms of the semiperimeter and the difference between it and the remaining side, and divide the sum by 2, the quotient is the logarithmic tangent of half the angle opposite to the latter side.

Ex. In a triangle, given the sides a = 679, b = 537, and c = 429; to find the angles.

Log. 143.5 = 2.156852	Log. 822.5 = 2.915136
Log. 285.5 = 2.455606	Log. 143.5 = 2.156852
Co-Log. 679 = 7.168130	Co-Log. 537 = 7.270026
Co-Log. 537 = 7.270026	Co-Log. 429 = 7.367543
2 | 19.050614	2 | 19.709557
Log. Sin. 19° 35′ = 9.525307	Log. Cos. 44°17′½ = 9.854779

Log. 143.5 = 2.156852	
Log. 393.5 = 2.594945	Ans.
Co-Log. 822.5 = 7.084864	A = 88° 35′
Co-Log. 285.5 = 7.544394	B = 52° 15′
2 | 19.381055	C = 39° 10′
Log. Tan. 26°7′½ = 9.690528	180° 0′ Proof.

174. The preceding rules are more than sufficient for solving all the cases of plane triangles, whether right-angled or oblique-angled. The solution of the cases of the former, to angles within the limits of Table VII., may be effected by means of Table B.

Ex. In a triangle ABC, right-angled at C, given the side c = 960, and the angle B = 56° 15′; to find the rest.

Cou.	Dist.	Lat.	Dep.	Ans.
5	900	500.01	748.32	A = 33° 45′
pts.	60	33.33	49.89	a = 533.34
	960	533.34	798.21	b = 798.21

175. *NAPIER'S Rules of the Five Circular Parts.*
Rule. 1. The logarithmic sine of the mean part is equal to the sum of the logarithmic tangents of the adjacent parts, minus 10.

Rule 2. The logarithmic sine of the mean part is equal to the sum of the logarithmic cosines of the remote parts, minus 10.

Note. The circular parts of a right-angled spherical triangle, are the legs, the complement of the hypotenuse, and the complements of the oblique angles.

Ex. 1. In a spherical triangle ABC, right-angled at C, given $A = 75° 36'$, and $a = 64° 3'$; to find the rest.

Log. cot. 75° 36′ = 9.409521	Log. cos. 75° 36′ = 9.395658
Log. tan. 64 3 = 10.312781	Log. tan. 31 51 = 9.793256
Log. sin. 31 51 = 9.722302	Log. cot. 68 11 = 9.602402
Log. cot. 68 11 = 9.602402	Log. cos. 31 51 = 9.929129
Log. tan. 64 3 = 10.312781	Log. cos. 64 3 = 9.641064
Log. cos. 34 39 = 9.915183	Log. cos. 68 11 = 9.570193

$B = 34° 39'$, $b = 31° 51'$, $c = 68° 11'$. Ans.

176. *Given the three sides of an oblique-angled spherical triangle, to find the angles.* To the logarithmic sines of the differences between the semiperimeter, and any two sides, add the arithmetical complements of the logarithmic sines of the semiperimeter and the difference between it and the remaining side; half the sum is the logarithmic tangent of half the angle opposite to the latter side.

Ex. In an oblique-angled spherical triangle, given the sides $a = 16° 24'$, $b = 161° 36'$ and $c = 157° 22'$; to find the angles.

Log. sin. 4°35′ = 8.902596	Log. sin. 152°47′ = 9.660255
Log. sin. 11 49 = 9.311289	Log. sin. 11 49 = 9.311289
Co-Log.sin.169 11 = 0.726612	Co-Log sin.169 11 = 0.726612
Co-Log.sin.152 47 = 0.339745	Co-Log.sin.4 35 = 1.097404
2 | 19.280242	2 | 20.795560
Log. tan. 23 35 = 9.640121	Log.tan.68 11 = 10.397780

$A = 47° 10'$, $B = 136° 22'$, $C = 88° 34'$. Ans.

177. *Given the three angles of an oblique-angled spherical triangle, to find the sides.* To the logarithmic cosine of half the sum of the three angles, (taken with a contrary sign,) and the logarithmic cosine of the difference between it and one of the angles, add the arithmetical complements of the logarithmic cosines of the differences between the other angles and half the sum of the three angles; divide the sum of these logarithms by 2, and the quotient is the tangent of half the side opposite the former angle.

Ex. In an oblique-angled spherical triangle, given the

angles A = 56° 52′, B = 72° 13′, and C = 107° 47′; to find the sides.

—Log. cos. 118° 26′	=9.677731	—Log. cos. 118° 26′	=9.677731
Log. cos. 61 34	=9.677731	Log. cos. 10 39	=9.992454
Co-Log. cos. 46 13	=0.159936	Co-Log. cos. 61 34	=0.322269
Co-Log. cos. 10 39	=0.007516	Co-Log. cos. 46 13	=0.159936
	2 | 19.522944		2 | 20.152390
Log. tan. 30 0	= 9.761472	Log. tan. 50 0	=10.076195

$a = 60°$, $b = 80°$, $c = 100°$. Ans.

NAPIER'S Analogies, and the Rule of Four Sines.

178. *Given two sides and the contained angle; or two angles and the interjacent side; to find the rest.*

Rule 1. The logarithmic tangent of half the sum of any two angles is equal to the sum of the logarithmic cosine of half the difference of the opposite sides, the logarithmic secant of half their sum, and the logarithmic cotangent of half the third angle, minus 20.

Rule 2. The logarithmic tangent of half the difference of any two angles is equal to the sum of the logarithmic sine of half the difference of the opposite sides, the logarithmic cosecant of half their sum, and the logarithmic cotangent of half the third angle, minus 20.

179. Rule 3. The logarithmic tangent of half the sum of any two sides is equal to the sum of the logarithmic cosine of half the difference of the opposite angles, the logarithmic secant of half their sum, and the logarithmic tangent of half the third side, minus 20.

180. Rule 4. The logarithmic tangent of half the difference of any two sides is equal to the sum of the logarithmic sine of half the difference of the opposite angles, the logarithmic cosecant of half their sum, and the logarithmic tangent of half the third side, minus 20.

181. Rule 5. *To find a side.* The logarithmic sine of any side is equal to the sum of the logarithmic sines of the opposite angle, and an adjacent side, minus the logarithmic sine of the angle opposite to the latter.

182. Rule 6. *To find an angle.* The logarithmic sine of any angle is equal to the sum of the logarithmic sines of the opposite side, and an adjacent angle, minus the logarithmic sine of the side opposite to the latter.

Ex. 1. In a spherical triangle, given $a = 59°\ 13'$, $b = 81°\ 17'$ and C = 125 36′; to find the rest.

Log. cos. 11° 2′	= 9.991897	Log. sin. 11° 2′	= 9.281897
Co. Log. cos. 70 15	= 0.471190	Co-Log. sin. 70 15	= 0.026329
Log. cot. 62 48	= 9.710904	Log. cot. 62 48	= 9.710904
Log. tan. 56 11	= 10.173991	Log. tan. 5 58	= 9.019130

Log sin. 125° 36′ = 9.910144	
Log. sin. 81 17 = 9.994955	Ans.
19.905099	B = 62° 9′
Log. sin. 62 . 9 = 9.946538	A = 50 13
Log. sin. 114 38 = 9.958561	c = 114 38

Ex. 2. In a spherical triangle given $A = 66° 40'$, $B = 41° 32'$, and $c = 60° 51'$; to find the rest.

Log cos. 12°34′ = 9.989469	Log. sin 12°34′ = 9.337610
Co-Log. cos 54 6 = 0.231827	Co-Log. sin. 54 6 = 0.091493
Log. tan. 30 25½ = 9.768847	Log tan. 30 25½ = 9.768847
Log. tan. 44 21 = 9.990143	Log. tan. 8° 57.85 = 9.197950
Log. sin. 66 40 = 9.962945	
Log. sin. 60 51 = 9.941187	Ans.
19.904132	a = 53° 18′.85
Log sin. 53° 18.85 = 9.904132	b = 35 23.15
Log. sin. 90° = 10.000000	C = 90 0

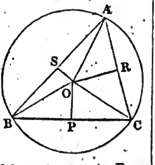
Note. The preceding rules are more than sufficient for solving all the cases of spherical triangles, whether right-angled or oblique-angled.

SECTION III.—INVESTIGATION OF THE RULES.

I. PLANE TRIGONOMETRY.

183. The fundamental proposition in Trigonometry is, that, *the sides of a plane triangle are to each other as the sines of the opposite angles.* The following is perhaps the most convincing demonstration to a learner. About any triangle ABC, (fig 1.) describe the circle ABC, and from the centre O, draw the straight lines OA, OB, OC, to the angles, and the perpendiculars, OP, OR, OS, to the sides. Then, because the angle BOP = ½ BOC, therefore, the angle BOP = BAC, (Euclid, 20th of 3d book,) and BP or ½ BC = sin BAC. In like manner, ½ BA = sin BCA; and ½ CA = sin ABC. Hence, to the radius of the circumscribing circle, $\frac{1}{2}a = \sin A$, $\frac{1}{2}b = \sin B$, and $\frac{1}{2}c = \sin C$. Therefore, $\frac{1}{2}a : \frac{1}{2}b :: \sin A : \sin B$, or $a : b :: \sin A : \sin B$; also, $a : c :: \sin A : \sin C$; and $b : c :: \sin B : \sin C$. On this proposition, are founded the

fig. 1.



two rules in art. 171. For, log sin B = co-log a + log b + log sin A — 10, (art. 102); and since, sin A : sin B : : a : b, therefore, also, log b = log a + co-log sin A + log sin B — 10 (art. 102).

184. From this proposition a great many useful corollaries may be derived. If the angle C in fig. 1, be considered a right angle, then sin C = radius, and A + B = 90°; whence the preceding proportions are changed into the following, viz.: $a : b$: : cos B : sin B, or rad. : tan B : : $a : b$; also, rad. : sin A : : $c : a$, or rad. : cos B : : $c : a$; whence, rad. : sec A : : $a : c$, or rad. : cosec B : : $a : c$.

The translation of these analogies into popular language, is, that in any right-angled triangle, the hypotenuse is to either leg, as radius to the sine of the angle opposite that leg, or to the cosine of the angle adjacent to that leg; next, either leg is to the other, as radius to the tangent of the angle opposite the latter; and, lastly, either leg is to the hypotenuse, as radius is to the secant of the contained angle, or the cosecant of the other acute angle.

185. The next important step in Trigonometry, is to find an expression for the sine of the sum and difference of two arcs or angles, in terms of the sines and cosines of those arcs or angles. The following solution of this problem is similar to a very ingenious one given by Dr Forbes, in his "Theory of the Differential Calculus," at section 8, on the Circular Theorem. Produce BC, any side of a triangle ABC, (figs. 2 and 3,) to the point E, and draw CD perpendicular to AB.

fig. 2.

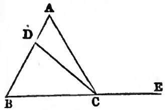
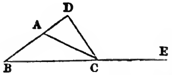


fig. 3.



Then, because (183) CB : BA : : sin A : sin ACB, or sin ACE, or sin (A ± B), therefore,

$$\sin (A \pm B) = \frac{BA}{BC} \cdot \sin A, \text{ or}$$

$$\sin (A \pm B) = \sin A \cdot \frac{BD \pm DA}{BC} \quad (1)$$

But (184) BD : BC : : cos B : rad., therefore,

$$\frac{BD}{BC} = \frac{\cos B}{\text{rad.}} \quad (2)$$

And DC : BC : : sin B : rad., therefore,

$$\frac{DC}{BC} = \frac{\sin B}{\text{rad.}} \quad (3)$$

Again, DA : DC : : cos A : sin A, therefore,

$$\frac{DA}{DC} = \frac{\cos A}{\sin A} \quad (4)$$

Whence, from (3) and (4), denoting rad. by R, we have

$$\frac{DA}{DC} \cdot \frac{DC}{BC} = \frac{\cos A}{\sin A} \cdot \frac{\sin B}{R} \text{ or } \frac{DA}{BC} = \frac{\cos A \sin B}{R \sin A} \quad (5)$$

Now, from (1), (2), and (5), we have

$$\sin (A \pm B) = \sin A \left(\frac{\cos B}{R} \pm \frac{\cos A \sin B}{R \sin A} \right)$$

$$\text{or, } \sin (A \pm B) = \frac{\sin A \cos B \pm \cos A \sin B}{R}$$

and, when $R = 1$, it becomes formula I., viz.:

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B.$$

186. In the same manner, we might find an expression for the cosine of the sum and difference of two arcs or angles, in terms of the sines and cosines of those arcs or angles. We prefer, however, the following method: since $\frac{1}{2}\pi$ denotes, in reference to arcs, the quadrant of a circle; and, in reference to angles, the right angle; if we substitute $\frac{1}{2}\pi - A$ for A, in formula I., we have

$$\sin \left(\tfrac{1}{2}\pi - A \pm B\right) = \sin \left(\tfrac{1}{2}\pi - \left\{ A \mp B \right\}\right) =$$

$$\sin \left(\tfrac{1}{2}\pi - A\right) \cos B \pm \cos \left(\tfrac{1}{2}\pi - A\right) \sin B.$$

$$\text{But since, } \sin \left(\tfrac{1}{2}\pi - \left\{ A \mp B \right\}\right) = \cos (A \mp B);$$

and, $\sin (\frac{1}{2}\pi - A) = \cos A$; also, $\cos (\frac{1}{2}\pi - A) = \sin A$; we have, by substituting these values in the preceding expression, formula II., viz.:

$$\cos (A \mp B) = \cos A \cos B \pm \sin A \sin B.$$

187. It might seem from the preceding demonstration of these fundamental formulæ of Analytical Trigonometry, that the symbols A and B are restricted to those arcs whose sum is less than a semicircle. But when it is considered that the sine of any arc A, is also the sine of $\pi - A$, or of $2n\pi + A$, or of $(2n + 1)\pi - A$; and that the cosine of any arc A, is also the cosine of $2n\pi \pm A$; where n may be zero or any number whatever; it will appear that the demonstration is sufficiently general, without encumbering it with the consideration of arcs whose sum is greater than a semicircle.

188. If we take the sum and difference of the double formulæ I. and II., we shall obtain the four following formulæ, III., IV., V., and VI., viz.:

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B$$
$$\sin(A+B)-\sin(A-B)=2\cos A\sin B$$
$$\cos(A-B)+\cos(A+B)=2\cos A\cos B$$
$$\cos(A-B)-\cos(A+B)=2\sin A\sin B$$

189. If we suppose $A=x+y$, and $B=x-y$; we shall have $x=\frac{1}{2}(A+B)$ and $y=\frac{1}{2}(A-B)$; hence, by substituting in the four preceding formulæ x and y, instead of A and B, (as being equally true of the former and the latter) and then substituting the values of x and y just found in terms of A and B, in these new expressions, we shall obtain the four following formulæ, VII., VIII., IX., and X., viz.:

$$\sin A+\sin B=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$
$$\sin A-\sin B=2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$
$$\cos B+\cos A=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$
$$\cos B-\cos A=2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$

190. If we divide the members of formula VII., by the corresponding members of VIII.; and those of IX., by those of X., and reduce the second members of each resulting equation to their most abridged form, we shall obtain the four following formulæ, XI., XII., XIII., and XIV., viz.:

$$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$$
$$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\cot\frac{1}{2}(A-B)}{\cot\frac{1}{2}(A+B)}$$
$$\frac{\cos B+\cos A}{\cos B-\cos A}=\frac{\cot\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$$
$$\frac{\cos B+\cos A}{\cos B-\cos A}=\frac{\cos\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$$

191. The derivation of formulæ in this manner might be carried to a great extent, and there is no branch of the Mathematics more fascinating from its simplicity and beauty, and more useful in its application to the various investigations of Mechanical Philosophy and Astronomy; but we must hasten to deduce the practical rules which depend on some of the preceding formulæ. In any triangle ABC,

(fig. 1.) we have seen that $a : b :: \sin A : \sin B$, (183), whence, $a + b : a - b :: \sin A + \sin B : \sin A - \sin B$, (see Thomson's Euclid, Appendix to Book V., cor. to Prop. X.) Therefore

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}, \text{ whence,}$$

by formula XI., we obtain formula XV., viz.:

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

From this formula, we derive the following analogy, viz.:

$$a + b : a - b :: \tan \tfrac{1}{2}(A+B) : \tan \tfrac{1}{2}(A-B);$$

which, expressed in words, is as follows: *the sum of any two sides of a plane triangle, is to their difference, as the tangent of half the sum of the angles opposite to those sides, is to the tangent of half their difference.* This is the foundation of rule 1, art. 172. For $\log \tan \frac{1}{2}(A - B) = \log \tan \frac{1}{2}(A + B) + \text{colog}(a + b) + \log (a - b) - 10$, art. 102.

192. To demonstrate rule 2, art. 172, we may investigate some additional formulæ. Thus, if in formulæ VII., or VIII., and in formulæ IX., and X., (art. 189.) we make $B = 0$, we shall have, (since $\sin 0 = 0$, and $\cos 0 = 1$) the three following formulæ, XVI., XVII., and XVIII., viz.:

$$\sin A = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A$$
$$1 + \cos A = 2 \cos^2 \tfrac{1}{2} A$$
$$1 - \cos A = 2 \sin^2 \tfrac{1}{2} A$$

193. Since the arc A, in these formulæ, may be of any magnitude, we may suppose it increased by B, in formula XVI., and we shall have

$$\sin (A + B) = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A + B)$$

Now, by dividing the members of this formula by the corresponding members of formulæ VII. and VIII., art. 189, and reduce the second members of each resulting equation to their most abridged form, we shall obtain the two following formulæ, XIX. and XX., viz.:

$$\frac{\sin (A+B)}{\sin A + \sin B} = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)}$$

$$\frac{\sin (A+B)}{\sin A - \sin B} = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)}$$

194. Again, since by art. 183, we have, in any triangle ABC,

$$\frac{\sin A}{\sin C} = \frac{a}{c}, \text{ and } \frac{\sin B}{\sin C} = \frac{b}{c},$$

by taking the sum and difference of the corresponding members of these equations, and substituting $\sin(A + B)$ instead of $\sin C$, in the denominators of the first members of the resulting equations, we shall have the two following formulæ, viz.:

$$\frac{\sin A + \sin B}{\sin(A + B)} = \frac{a + b}{c}$$

$$\frac{\sin A - \sin B}{\sin(A + B)} = \frac{a - b}{c}$$

But, by inverting the members of formulæ XIX. and XX., art. 192, and comparing them with these, we obtain the two following formulæ, XXI. and XXII., viz.:

$$\frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} = \frac{a + b}{c}$$

$$\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = \frac{a - b}{c}$$

From the first of these formulæ, we derive the following analogy, viz.:

$$\cos \tfrac{1}{2}(A - B) : \cos \tfrac{1}{2}(A + B) :: a + b : c,$$

which, expressed in words, is as follows: *the cosine of half the difference of any two angles of a triangle, is to the cosine of half their sum, as the sum of the two sides opposite those angles is to the third side.* This is rule 2, art. 172, when translated into logarithmic language; thus, $\log c = \log \cos \frac{1}{2}(A + B) + \text{colog} \cos \frac{1}{2}(A - B) + \log(a + b) - 10$.

195. To demonstrate the rules in art. 173, we have, by referring to figs. 2 and 3, art. 185, $BC^2 = BD^2 + CD^2$ (1), see Euclid, 47th Prop. of B. I. But $CD^2 = AC^2 - AD^2$ (2), and $DB^2 = (AB \mp AD)^2 = AB^2 \mp 2AB.AD + AD^2$ (3), see Euclid, 4th prop. and cor. 1st to 7th Prop. B. II., Thomson's Edition. Therefore, by substituting in equation (1), the values of CD^2 and DB^2 obtained in equations (2) and (3), we have, $BC^2 = AB^2 \mp 2AB.AD + AD^2$

$+ AC^2 - AD^2$; whence, $BC^2 = AB^2 \mp 2AB.AD + AC^2$ (4). But, by art. 184, rad : $\pm \cos A$: : AC : AD, whence, making rad. = 1, AD $= \pm$ AC cos A (5); and substituting this value of AD in (4), we have

$$BC^2 = AB^2 - 2AB.AC \cos A + AC^2;$$

or, adopting the usual symbols for the sides, we obtain formula XXIII., viz.: $a^2 = b^2 + c^2 - 2bc \cos A$; whence, by transposition and division, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

From this formula, the angles of a triangle may be found, when the sides are given, by a table of natural sines.

196. In formula XXIII., by adding $-2bc + 2bc$, or nothing, to the second member, we have

$$a^2 = b^2 - 2bc + c^2 + 2bc\,(1 - \cos A)$$

which, by transposition, changing signs, &c., becomes

$$2bc\,(1 - \cos A) = a^2 - (b - c)^2 = (a + b - c)\,(a - b + c)$$

See Euclid, Prop. 5th, cor. 1st. B. II. In order to abridge this formula, put $a + b + c = 2s$; then, $a + b - c = 2\,(s - c)$, and $a - b + c = 2\,(s - b)$; now, by substituting these values, and refering also to formula XVIII., art. 192, we have $4bc \sin^2 \frac{1}{2} A = 4\,(s - b)\,(s - c)$; whence, by dividing by $4bc$, and extracting the square root, we obtain formula XXIV., viz.:

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s - b)\,(s - c)}{bc}}.$$

197. If to the second member of formula XXIII., we had added $2bc - 2bc$, or nothing, in such a manner as to complete the square of $a + b$, instead of that of $a - b$, we should, by a process similar to the preceding, have obtained formula XXV., viz.:

$$\cos \tfrac{1}{2} A = \sqrt{\frac{s\,(s - a)}{bc}}.$$

198. Now by dividing the members of formula XXIV., by the corresponding members of XXV., we obtain formula XXVI., viz.:

$$\tan \tfrac{1}{2} A = \sqrt{\frac{(s - b)\,(s - c)}{s\,(s - a)}}.$$

199. If the three formulæ just demonstrated, which are the foundation of the rules in art. 173, be translated into popular language, it will be seen that they are as applicable to the angles B and C of any triangle ABC, as they are to the angle A. We translate the first both into popular and logarithmic language, leaving the other two as exercises for the learner. Thus, *the sine of half any angle of any triangle is equal to the square root of the quotient obtained by dividing the product of the differences between the semi-perimeter, and each of the sides containing that angle, by the product of those sides.* Whence, also, log sin $\frac{1}{2}$ A = $\frac{1}{2}$ $\left\{\log(s-b) + \log(s-c) + \text{colog}\, b + \text{colog}\, c\right\}$. It may be remarked here, that there is an apparent omission in this logarithmic formula, which must be explained. Whenever the complements of logarithms are added in any formula, instead of the logarithms themselves being subtracted, we ought to deduct as many tens from the amount as there were complements employed, which, in the preceding case, would be 2 tens. But as the logarithmic radius of the tables in common use is 10, and were it introduced into formula XXIV.; instead of being only understood, it would appear as a divisor to sin $\frac{1}{2}$ A, or a multiplier to its equivalent; and when brought under the radical sign, its square, 20, would be added to the sum of the logarithms of the quantities under that sign; so, the omission of the square of the logarithmic radius compensates for the omission of the 20 in the above logarithmic formula, the one being added and the other subtracted; and thus we obtain in practice the logarithmic sine of the Tables.

200. To demonstrate some rules contained in the preceding sections, and to lead to some useful applications of Trigonometry, we resume the consideration of the necessary formulæ. Thus, by multiplying the corresponding members of formulæ XXIV. and XXV. together, doubling each member of the product, and referring to formula XVI., we have

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

201. Now, multiplying both members of this formula by $\frac{1}{2}bc$, we have

$$\tfrac{1}{2}bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}.$$

But (184) R : sin A :: b : CD, figs. 2 and 3; therefore, CD $= b \sin A$, when R = 1; and denoting CD, the perpendicular of any triangle by p, we have the two following formulæ, XXVII and XXVIII, viz.:

$$\tfrac{1}{2}bc \sin A = \tfrac{1}{2}pc = \text{area of ABC}.$$

$$\text{area of ABC} = \sqrt{s(s-a)(s-b)(s-c)}.$$

202. The translation of these formulæ (which are the foundation of the rules in art. 143,) into popular and logarithmic language, is as follows: the area of any triangle is equal to half the product of the base and altitude or perpendicular; to half the product of any two sides, and the sine of their contained angle; or, to the square root of the continued product of the semiperimeter and the differences between it and the three sides. Hence, log area $=$ log c $+$ log p $+$ log $\frac{1}{2}$; log area $=$ log b $+$ log c $+$ log sin A $-$ 10 $+$ log $\frac{1}{2}$; and log area $= \frac{1}{2}\left\{\log s + \log (s-a) + \log (s-b) + \log (s-c)\right\}$.

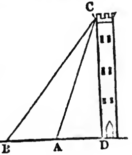
203. As it distinctly appears from the cor. to the 4th Prop. of the 4th Book of Euclid, that the area of a triangle is equal to the product of the semiperimeter and the radius of the inscribed circle; denoting this radius by r, we have from formula XXVIII. the following expression:

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

II.—MENSURATION OF HEIGHTS AND DISTANCES.

204. In the determination of the elevations of objects above the ground, some of the simplest cases of Trigonometry only are necessary, unless from the obliquity of the line assumed as a base, or the investigation of any other necessary corrections pointed out by practice, a greater degree of accuracy be acquired than is attainable by the common rules. The height of an *accessible* object as CD, fig. 4, above a horizontal plane BAD, may be found by measuring the *distance* AD of the foot of the object, from A any point assumed as a *station*, at which the *angle of elevation* DAC, is determined by a quadrant or theodolite, and proceeding according to art. 184. Thus, denoting DAC the *angle of elevation* by E, the *distance* AD by D, and the required *height* by H, we have R : tan E : : D : H; whence, when R $=$ 1, we have H $=$ D tan E, or log H $=$ log tan E $+$log D $-$ 10; also, in words, the logarithm of the height

fig. 4.



the height of the mountain, supposing the deck of each vessel to be 30 feet above the level of the sea.

$$\begin{array}{rl} \text{Log } 5280 = & 3.722634 \\ \text{Log sin } 46^\circ\ 10' = & 9.858151 \\ \text{Log sin } 34\ \ 50 = & 9.756782 \\ \text{Log cosec } 11\ \ 20 = & 10.706601 \\ \hline & 34.044168 \\ \text{Log } 11070 = & 4.044168 \end{array}$$ Ans. 11100 feet.

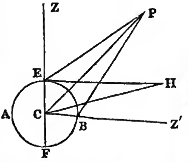
Or, thus;

$$\begin{array}{rl} \text{Nat. cot } 34^\circ\ 50' = & 1.4370268 \\ \text{Nat. cot } 46\ \ 10 = & .9600829 \\ \hline & .4769439 \end{array}$$

whence, $\frac{5280}{.4769439} = 11070$; now, $11070 + 30 = 11100$ feet.

207. The determination of the *distances of the Sun and Moon*, are very simple applications of the principles of Trigonometry. Thus, let AEBF, fig. 5, represent a section of the earth, through its centre C; H, the place of a heavenly body, when on the horizon to a spectator at E; then, the angle EHC, under which the semi-diameter of the earth would be seen by a spectator placed at the heavenly body H, is called its *Horizontal Parallax*. Now, denoting the angle EHC by H, the *distance* CH by D, and the *radius* of the earth CE by T, by art. 184, we have, R : sin H :: D : T; whence, by making R = 1, we have, D sin H = T, or $D = \frac{T}{\sin H}$; and log D = log T + 10 — log sin H.

fig. 5.



Ex. 1. The moon's mean horizontal parallax is about 57', and the earth's mean diameter about 7912 miles; required the moon's mean distance from the earth.

$$\begin{array}{rl} 10 + \text{Log } 3956 = & 13.597256 \\ \text{Log sin } 57' = & 8.219581 \\ \hline \text{Log } 238603 = & 5.377675 \end{array}$$ Ans. 238603 miles.

Ex. 2. The sun's mean horizontal parallax is about 8".5776; required the sun's mean distance from the earth.

$$\begin{aligned}
\text{Log sin } 1' &= 6.463726 \\
\text{Co-Log } 60'' &= 8.221849 \\
\text{Log } 8''.5776 &= \underline{0.933366} \\
& \quad 15.618941 \\
\text{Log sin } 8''.5776 &= 5.618941 \text{ nearly} \\
10 + \text{Log } 3956 &= \underline{13.597256} \\
\text{Log } 95129500 &= 7.978315 \qquad \text{Ans. } 95129500 \text{ miles.}
\end{aligned}$$

208. To determine the *horizontal parallax* itself, is a very interesting problem. Its approximate solution is as follows: In fig. 5, let E be the place of a spectator on the surface of the earth, Z the zenith, P a heavenly body, EH the horizon, and C the centre of the earth. The difference between the zenith distances of the body ZEP, ZCP, as seen from the surface, and from the centre of the earth, which is the same as the difference of their altitudes in the same circumstances, is called the *parallax in altitude.* The angle EPC represents this parallax, and it is always less than the *horizontal parallax* EHC, diminishing as the body approaches the zenith, where it vanishes altogether Now, in the triangle ECP, we have, by art. 183, sin EPC : sin CEP : : EC : CP; and denoting the angle EPC by P, the angle ZEP by Z, and EC, CP, by T and D as before, and recollecting that sin ZEP = sin CEP, we have, sin P : sin Z : : T : D, whence, D sin P = T sin Z, and $D = \frac{T \sin Z}{\sin P}$; whence, by art. 207, we have, $\frac{T}{\sin H} = \frac{T \sin Z}{\sin P}$, or sin P = sin H sin Z; but as the sines of small arcs are to each other nearly as the arcs themselves; and as the angles P and H, are small for all the heavenly bodies, being in fact greatest for the moon itself, we have, P = H sin Z. From this we see that the *parallax in altitude is equal to the product of the horizontal parallax, and the sine of the zenith distance, or the cosine of the altitude.*

209. Again, if E and B be the places of two observers, under the same meridian, Z and Z′, their zeniths, the angle ECB, will be the difference of their latitudes, L and L′, and the angles ZEP, Z′EP, denoted by Z and Z′ the zenith distances of the body observed at the same instant; now, in the quadrilateral ECBP, three angles are known, for the angles PEC, PBC, are the supplements of the zenith distances; whence, the angle EPB is known, being the explement of the sum of those three angles. Therefore, denoting the angle EPB by Q, we have $Q = Z + Z' - (L \pm L')$.* But, denoting the angle EPC as before,

* The upper or lower sign is to be used, according as the latitudes are of different names, or of the same name.

by P, and BPC by P′, we have, also, $P' = H \sin Z'$; whence, $P + P' = Q = H \sin Z + H \sin Z'$; and by formula VII., art. 189, we have,

$$H = \frac{Q}{\sin Z + \sin Z'} = \frac{Z + Z' - (L \pm L')}{2 \sin \frac{1}{2}(Z + Z') \cos \frac{1}{2}(Z - Z')}$$

Ex. Lacaille and Wargentin made the following observations at the Cape of Good Hope, and Stockholm, respectively, on the 5th of Nov., 1751.

At the Cape, zen. dist. ☽'s north limb, 56° 39′ 40″.
At Stockholm, zen. dist. ☽'s north limb, 38° 4′ 52″.
Latitude of the Cape, 33° 56′ 3″ S.
Latitude of Stockholm, 59° 20′ 31″ N.

Here, $Q = Z + Z' - (L + L') = 1° 27' 58'' = 87'.966$; $Z + Z' = 94° 44' 32''$, and $Z - Z' = 18° 34' 48''$; whence,

$$\begin{aligned} \text{Log } 87'.966 &= 1.944318 \\ \text{Co-Log } 2 &= 9.698970 \\ \text{Co-Log } \sin 47° 22' 16'' &= 0.133259 \\ \text{Co-Log } \cos 9\ 17\ 24 &= \underline{0.005863} \\ \text{Log } 60''.5913 &= 1.782410 \end{aligned}$$

Consequently, the moon's horizontal parallax, according to these observations, is 60′ 35½″; this quantity, however, varies according to her distance from the earth, between the extremes 61′ 24″, and 53′ 48″.

210. In the measurements of heights or distances, by means of angular instruments, it becomes important to know under what circumstances any small error of observation will produce the smallest effect on the calculated result. This will be most easily ascertained by reference to Taylor's Theorem; thus, in the triangle CAD, fig. 4, we have, by art. 204, $H = D \tan E$; whence, denoting any small error in the angle E by e, and the corresponding error in H by h, we have,

$$H + h = D \tan (E + e) =$$

$$D \left\{ \tan E + \frac{d \tan E}{dE} e + \frac{d^2 \tan E}{2dE^2} e^2 + \&c. \right\};$$

$$\text{whence, } h = D \left\{ e \sec^2 E + e^2 \sec^2 E \tan E + \&c. \right\};$$

$$\text{but, since } D = \frac{H}{\tan E}, \text{ and } \frac{\sec^2 E}{\tan E} = \frac{2}{\sin 2E};$$

$$\text{therefore, } h = H \left\{ \frac{2e}{\sin 2E} + \frac{2e^2 \tan E}{\sin 2E} + \&c. \right\}.$$

Now, it is plain that h will be a *minimum* when the denominator of the fractions within the brackets is a *maximum*; this will be the case, when $\sin 2\,E = 1$, or when $E = 45°$. Consequently, the height will be least affected by any error of observation, when the angle of elevation is 45°, that is, when the height is equal to the distance. To these remarks, it may be added, that when the angle of elevation is 30°, the height is half the distance; and when it is 60°, the height is double the distance.

Ex. Supposing an error of 1 minute to be made in observing an angle of elevation of 45°, what error would be produced in the computed height?

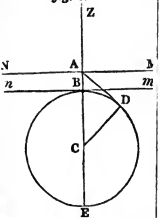
Here it is only necessary to compute the first term of the preceding series for the value of h; because by introducing the proper powers of R, the radius, to render the terms homogeneous, they become so small as to be useless in the result; hence, we have,

$$h = H \times \frac{2e}{R} = \frac{2 \times 1' \times H}{3438'} = \frac{H}{1719}.$$

Thus, we see that an error of 1' produces an error in the computed height of only a 1719th part of itself, when the angle of elevation is 45°. At any other angle the error is greater, in proportion as the angle recedes from 45°.

211. Connected with the practice of surveying, is the determination of the *difference between the true and the apparent levels* of objects at the earth's surface. To investigate this correction; let BDE, fig. 6, be a central section of the earth; then, the arc DB represents the true level between any two points on its surface, and the straight line DA the apparent level; the straight line or height AB, represents the difference between the true and apparent level, and the arc DB, or its tangent DA, which differ very little from each other in practice,* the distance between the points. Now, denoting CD or CB, the radius of the earth, which we here reckon to be 3960 miles, by T, the height AB by h, and the distance DA by d, we have, by the Pythagorean theorem,

fig. 6.



* For the length of one degree, which is about 70 miles on the earth's surface, this difference is only about a ten-thousandth part.

$(T + h)^2 - T^2 = d^2$; whence, $2Th + h^2 = d^2$, and $d = \sqrt{2Th + h^2}$. To find the value of h, requires the resolution of this quadratic equation; but as the term h^2 is, in practice, always an extremely minute fraction compared with the term $2Th$, we may, in general, omit the former, and we shall have $2Th = d^2$. Now, as h is generally given or required in feet, and T and d in miles, it will be necessary to render the equation homogeneous; whence, $\frac{7920\,h}{5280} = d^2$, and $h = \frac{2}{3}d^2$. Thus, we find, *that the height or difference between the true and the apparent level of two points on the surface of the earth, whose distance is given in miles, is equal to the number of feet obtained by taking $\frac{2}{3}$ of that distance.* Thus, we obtain the following table of differences in inches for miles in distance:

Distances,	1,	2,	3,	4,	5,	6,	7,	8,	9,
Differences,	8,	32,	72,	128,	200,	288,	392,	512,	648.

These differences are given in inches to avoid fractions; but if we take the distances in *leagues*, and the differences in *fathoms*, we have the following table, which exhibits the law in a form more easily to be remembered:

Distances,	1,	2,	3,	4,	5,	6,	7,	8,	9,
Differences,	1.	4,	9,	16,	25,	36,	49,	64,	81.

These remarks explain arts. 138 and 141; and the following will explain art. 137, and the intermediate ones.

212. Since $h = \frac{2}{3}d^2$, we have $d = \sqrt{h + \frac{1}{2}h}$, or $d = \sqrt{1\frac{1}{2}h}$; which, in logarithms, gives $\log d = \frac{1}{2}(\log h + \log 1.5)$ see art. 137. Or, conversely, $\log h = 2\log d - \log 1.5 = 2\log d + \log 2 - \log 3$, see art. 140. Thus the distance due to a given height or difference of level may be found and conversely.

213. The mode of calculating the *Depression or Dip of the Horizon*, which depends on the preceding principles, may now be investigated. In fig. 6, the straight line MN, parallel to mn touching the circle at B, represents the true visible horizon, and AD the observed horizon from the point A. Hence, the observed zenith distance ZAD, of any body on the horizon, is always greater than the true zenith distance ZAM, or Z Bm, by the angle MAD, which is called the *Depression or Dip*, due to the height AB. This angle MAD = the angle ACD at the centre of the earth, subtending the arc or distance DB, because DAC is their common complement. Whence, rad : tan Dip : : T : d, or

$\tan \text{Dip} = \frac{d}{T}$; in logarithms, $\log \tan \text{Dip} = 10 + \log d - \log T$. Now, as d can always be very easily found, in miles, from h in feet, we consider this as one of the simplest rules for finding the *depression* that has hitherto been given. The arithmetical rule for finding d is as follows: *To the height add half of itself, and extract the square root of the sum, for the distance in miles.*

Ex. If the deck of a vessel be 24 feet above the level of the sea, what is the allowance for Dip?

Here $\sqrt{24 + 12} = \sqrt{36} = 6$ miles, and $T = 3956$;

$$\begin{aligned} \text{whence, } 10 + \log 6 &= 10.778151 \\ \text{Log } 3956 &= \underline{\ 3.599256} \\ \text{Log tan } 5' \, 12'' &= 7.180895 \end{aligned}$$

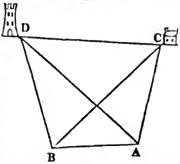
Thus, we find the allowance for Dip to be $5' \, 12''$; if $\frac{1}{10}$ of this allowance be deducted for *atmospherical refraction*, we shall obtain the Dip commonly inserted in nautical tables; see *Norie's Navigation*, Table V.

214. With regard to the determination of the allowance for *atmospherical refraction* itself, this depends so much on the state of the air in regard to temperature, that although it be trigonomically investigated, we avoid it here, merely remarking in the words of Professor Woodhouse, that "the *exact* formulæ of computation, with which foreign treatises abound, are in almost all cases, formulæ of curiosity—tools finer than the work to be done with them requires." Tables containing the dip for various altitudes allowing for the *mean effect* of refraction, are to be found in all collections for the use of the navigator and astronomer. "But these tabular dips," as Simms observes in his *Treatise on Mathematical Instruments*, "are at times found to differ so considerably from the truth, especially in tropical climates, that some experienced navigators have lately been induced to measure the actual dip by means of the *Dip Sector*, when any important determination has depended on the observation of altitudes." As an example, suppose from the deck of a vessel 24 feet above the level of the sea, the dip of the horizon had been observed to be $4' \, 41''$, by means of the instrument above mentioned; then, the difference between $5' \, 12''$ the dip due to the height, as found by calculation, art. 213, and the observed dip being $31''$; it is manifest that this difference can only be due to the refraction of the atmosphere, supposing the observation be free of error. Accordingly, the refraction, in this case, is $\frac{1}{10}$ part of the true dip; for, $\frac{31''}{5' \, 12''} \frac{31''}{312''} = \frac{1}{10}$ nearly. The allowance for

refraction, however, varies from $\frac{1}{7}$ to $\frac{1}{14}$ of the observ dip, according to the state of the atmosphere.

215. A very common and useful problem in plane Trig nometry, is to find the *distance between two inaccessi objects*. This is effected by the application of the rul already demonstrated, in the following manner: let C ai D, fig. 7, be the two inaccessible objects, and, of course, CD their distance. Take two stations, A and B, such, that the four points A, B, C, D, shall be in the same plane; and measure the distance AB. Measure, also, the angles BAC, BAD, at A, and the angles ABD, ABC, at B. Then, by rule 2, art. 171, find the side AC in the triangle ABC, and the side AD in the triangle ABD; and, since the angle CAD is known, being th difference between BAC and BAD, by rule 1, art. 17 find the angle ACD or ADC in the triangle ACD then, by rule 2, art. 172, or rule 2, art. 171, find the sid CD in the triangle ACD. This mode of solution require four separate calculations to determine CD, and no prac tical method has yet been proposed for diminishing thei number. The formulæ for these calculations may be com pactly exhibited in the following manner. Let the angl BAC = A, the angle BAD = O, the angle ABD = B, th angle ABC = P, and the angle CAD = Q. Also, let th side AB = a, AC = b, AD = c, and CD = x. Then, b art. 183, we have in the triangle ABC, sin ACB : sin AB(:: AB : AB; whence, sin (A + P) : sin P :: a : b; con sequently,

fig. 7.



$$b = \frac{a \sin P}{\sin (A + P)} \quad (1)$$

In like manner, from the triangle ABD, we obtain th formula

$$c = \frac{a \sin B}{\sin (B + O)} \quad (2)$$

But, by formula XXIII., we have, in the triangle ACD,

$$x^2 = b^2 + c^2 - 2bc \cos Q;$$

Now, by formula XVII., we have, by transposition,

$$\cos Q = 2 \cos^2 \tfrac{1}{2} Q - 1;$$

which being substituted in the preceding, gives

$$x^2 = b^2 + c^2 - 2bc\left\{2\cos^2\tfrac{1}{2}Q - 1\right\}$$

$$= b^2 + 2bc + c^2 - 4bc\cos^2\tfrac{1}{2}Q$$

$$(b+c)^2 - 4bc\cos^2\tfrac{1}{2}Q$$

$$= (b+c)^2\left\{1 - \frac{4bc}{(b+c)^2}\cos^2\tfrac{1}{2}Q\right\}.$$

Again, by putting $\sin^2 S = \dfrac{4bc}{(b+c)^2}\cos^2\tfrac{1}{2}Q$, or

$$\sin S = \frac{2\sqrt{bc}}{b+c}\cos\tfrac{1}{2}Q, \quad (3)$$

we have, by substitution in the preceding,

$$x^2 = (b+c)^2\left\{1 - \sin^2 S\right\} = (b+c)^2\cos^2 S,$$

whence, by extracting the square root, we have

$$x = (b+c)\cos S. \quad (4)$$

216. If we translate these four formulæ into logarithmic language, we shall have the four following equations, viz.:

I. $\log b = \log a + \log\sin P + \text{colog}\sin(A+P) - 10.$

II. $\log c = \log a + \log\sin B + \text{colog}\sin(B+O) - 10.$

III. $\log\sin S = \log 2 + \tfrac{1}{2}\log b + \tfrac{1}{2}\log c + \text{colog}(b+c)$
$+ \log\cos\tfrac{1}{2}Q - 10.$

IV. $\log x = \log(b+c) + \log\cos S - 10.$

Ex. In order to ascertain the distance between the two inaccessible objects C and D, a base AB of 2436 yards was measured, and the following angles were taken in the same plane, viz.: BAC = 115° 50′, BAD = 51° 12′, ABD = 120° 34′, ABC = 49° 50′ and consequently the angle CAD = 64° 38′; required the distance CD. Here, $a = 2436$, A = 115° 50′, O = 51° 12′, B = 120° 34′, P = 49° 50′, Q = 64° 38′, A + P = 165° 40′, and B + O = 171° 46′. Hence, by the preceding formulæ, we have,

$$\begin{aligned} \log 2436 &= 3.386677 \\ \log\sin 49^\circ 50' &= 9.883191 \\ \text{colog}\sin 165\ \ 40 &= \underline{0.606315} \\ \log 7519.4 &= \underline{3.876183} \end{aligned}$$

$$\begin{aligned}
\log 2436 &= 3.386677 \\
\log \sin 120^\circ\ 34' &= 9.935022 \\
\text{colog} \sin 171\ \ 46 &= \underline{0.844043} \\
\log 14646.8 &= \overline{4.165742} \\
\log 2 &= \underline{0.301030} \\
\tfrac{1}{2} \log 7519.4 &= 1.938092 \\
\tfrac{1}{2} \log 14646.8 &= 2.082871 \\
\text{colog}\ 22166.2 &= 5.654308 \\
\log \cos 32^\circ\ 19' &= \underline{9.926911} \\
\log \sin 53\ \ \ 9 &= \overline{9.903212} \\
\log \cos 53\ \ \ 9 &= \overline{9.777960} \\
\log 22166.2 &= \underline{4.345692} \\
\log 13293 &= \overline{4.123642}
\end{aligned}$$

Thus, the distance CD = 13293 yards, **Ans.**

217. By a process similar to that employed in art. 215, the two following formulæ might have been investigated instead of the two above, marked III. and IV.:

$$\text{V.}\ \log \tan T = \log 2 + \tfrac{1}{2} \log b + \tfrac{1}{2} \log c + \text{colog}\ (b - c) + \log \sin \tfrac{1}{2}\ Q - 10.$$

$$\text{VI.}\ \log x = \log (b - c) + \log \sec T - 10.$$

But, when $b = c$, $\sin S = \cos \frac{1}{2}\ Q$, and (art. 216),

$$\text{VII.}\ \log x = \log 2 + \log b + \log \sin \tfrac{1}{2}\ Q - 10$$

Ex. To exemplify formulæ V. and VI., suppose two ships to sail from the same port; the one N.E. by N., 84 miles, and the other S.E., 76 miles; required their distance from each other. Here the angle between the rhumb lines is 9 points, or $101^\circ\ 45' = Q$, (see Table VII. of the "*Practical Mathematician's Pocket Guide*,") $b{=}84$, and $c{=}76$; whence, we have

$$\begin{aligned}
\log 2 &= 0.301030 \\
\tfrac{1}{2} \log 84 &= 0.962140 \\
\tfrac{1}{2} \log 76 &= 0.940407 \\
\text{colog}\ 8 &= 9.096910 \\
\log \sin 4\tfrac{1}{2}\ \text{pts.} &= \underline{9.888185} \\
\log \tan 86^\circ\ 17'\tfrac{2}{3} &= \underline{11.188672} \\
\log \sec 86\ \ 17\tfrac{2}{3} &= \overline{11.189574} \\
\log 8\ \ \ \ \ \ &= \underline{0.903090} \\
\log 123.8 &= \overline{2.092664}
\end{aligned}$$

Consequently, the distance between the ships is 123.8 miles.

Note. When the given sides of a triangle are nearly equal to each other, formulæ III. and IV., art. 216, are to be used, when the given angle is large; and formulæ V. and VI., art. 217, when the given angle is small.

218. If a quadrilateral, such as ABDC, fig. 7, be inscribable in a circle, and its sides be given, any angle of it may be found. Thus, let $AB = a$, $BD = b$, $CD = c$, and $CA = d$; join AD, BC. Then, in the triangle ABC, denoting the angle BAC by A, from formula XXIII., art. 195, we have,

$$BC^2 = a^2 + d^2 - 2ad \cos A \quad (1)$$

In the triangle BCD, the angle $BDC = \pi - A$, whence, $\cos BDC = -\cos A$, and

$$BC^2 = b^2 + c^2 + 2bc \cos A \quad (2)$$

Now, putting the values of BC^2 in formulæ (1) and (2) equal to each other, transposing and abridging, we have

$$2(ad + bc) \cos A = a^2 + d^2 - b^2 - c^2;$$

$$\text{whence, } \cos A = \frac{a^2 + d^2 - b^2 - c^2}{2(ad + bc)} \quad (3)$$

Similar formulæ may be found for the other three angles.

219. By substituting in formula (1) the value of cos A found in formula (3), contracting or abridging the terms, and extracting the square root, we have

$$BC = \sqrt{\frac{(ac + bd)(ab + cd)}{ad + bc}} \quad (4)$$

A similar formula may be found for the other diagonal.

220. By finding the sum and difference of unity, and the value of cos A, formula (3), taking the products of the results, substituting $2s$ for $a + b + c + d$, recollecting that $(1 + \cos A)(1 - \cos A) = 1 - \cos^2 A = \sin^2 A$, and extracting the square root, we have

$$\sin A = \frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{ad + bc} \quad (5)$$

221. Denoting the radius of the circumscribing circle by R, it may be easily shown that

$$R = \tfrac{1}{4}\sqrt{\frac{(ab + cd)(ac + bd)(ad + bc)}{(s-a)(s-b)(s-c)(s-d)}} \quad (6)$$

222. Finally, the area of the quadrilateral ABDC being equal to that of the two triangles ABC, BCD, we have

$$\text{area } ABDC = \tfrac{1}{2}(ad \sin A + bc \sin D)$$
$$= \tfrac{1}{2}(ad + bc) \sin A; \text{ since, } \sin D = \sin A;$$

whence, by referring to formula (5), we have

$$\text{area ABDC} = \sqrt{(s-a)(s-b)(s-c)(s-d)} \quad (7)$$

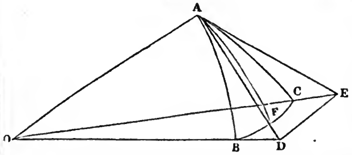
223. If the fourth side d of the quadrilateral figure ABDC, be supposed equal to zero, in each of the preceding formulæ relating to that figure, we shall obtain formulæ for triangles exactly the same as those which have been already demonstrated in art. 200, *et seq.* But here we must draw the subject of Plane Trigonometry to a close.

III.—SPHERICAL TRIGONOMETRY.

224. *Definitions.* We must premise here, that in Spherical, as well as in Plane Trigonometry, the three angles of any triangle are denoted by the letters A, B, C, and the three sides by a, b, c. In a plane triangle, the angles A, B, C, are plane angles contained by straight lines, and are measured by arcs of circles having the vertices for centres; but, in a spherical triangle, the angles A, B, C, are spherical angles, or angles on the surface of a sphere, contained each by two arcs of *great* circles of the sphere, (that is, by arcs of circles whose planes pass through the centre) and are measured by the inclination of their planes. Again, in a plane triangle, the sides a, b, c, are straight lines, measured by any unit of length whatever; but, in a spherical triangle, the sides a, b, c, are arcs of three *great* circles, which have their poles in the vertices of the angles of the triangle, and which, when produced till they intersect each other, form another triangle, called the *Supplementary* or *Polar triangle.*

225. The rules in Spherical Trigonometry are, in general, most easily derived from the following formula, which we now proceed to investigate. Let ABC, fig. 8, be a spherical

fig. 8.



riangle; O, the centre of the sphere, and OA, OB, OC, adii of the sphere. From the point A, draw the straight ines AD, AE, touching the sides AB, AC, of the triangle ABC at A, and meeting the radii OB, OC, produced in D, E; join DE. Then, (by 2d Def. 11th book, Euclid), the angle DAE is the measure of the spherical angle A, and he arc BC, or the side a, is the measure of the angle DOE. Now, by formula XXIII., art. 195, we have, in the triangle DAE,

$$DE^2 = DA^2 + AE^2 - 2DA.AE \cos A; \quad (1)$$

and, in the triangle DOE, we have,

$$DE^2 = DO^2 + OE^2 - 2DO.OE \cos a; \quad (2)$$

whence, by putting these values of DE^2 equal to each other, we have, formula (3) viz.:

$$DA^2 + AE^2 - 2DA.AE \cos A = DO^2 + OE^2 - 2DO.OE \cos a.$$

But, by the Pythagorean Theorem, we have

$$DO^2 = OA^2 + AD^2, \text{ and } OE^2 = OA^2 + AE^2;$$

whence, by substituting those values of DO^2 and OE^2, in (3), transposing and abridging, we have

$$DO.OE \cos a = DA.AE \cos A + OA^2; \quad (4)$$

But by art. 184, we have $DA = OA \tan c$, $AE = OA \tan b$, $DO = OA \sec c$, and $OE = OA \sec b$; whence, by substituting these values of DA, AE, DO, OE, in (4), and abridging, we have

$$\sec b \sec c \cos a = \tan b \tan c \cos A + 1; \quad (5)$$

But, since, by the principles of Plane Trigonometry, $\frac{1}{\sec b \sec c} = \cos b \cos c$, and $\tan b \cos b \tan c \cos c = \sin b \sin c$; by dividing both members of (5) by $\sec b \sec c$, we have formula XXIX., viz.:

$$\cos a = \sin b \sin c \cos A + \cos b \cos c.$$

In the same manner, or symmetrically, we may easily derive formulæ XXX. and XXXI., viz.:

$$\cos b = \sin a \sin c \cos B + \cos a \cos c$$

$$\cos c = \sin a \sin b \cos C + \cos a \cos b$$

226. From these three formulæ, the angles of a spherical triangle may be found when the sides are given; but the

operation would require a table of natural sines. To avoid this, by using formula II., art. 186, with the inferior sign, by substituting b and c for A and B, and transposing, we have

$$\cos b \cos c = \cos (b - c) - \sin b \sin c\,;$$

now, substituting this value of $\cos b \cos c$, in formula XXIX, and transposing and abridging, we have

$$\sin b \sin c\,(1 - \cos A) = \cos (b - c) - \cos a \quad (1)$$

Taking the value of $1 - \cos A$, from formula XVIII., art. 192, and of $\cos (b - c) - \cos a$, from formula X., art. 189, by substituting $b - c$ for B, and a for A, and putting these values in (1) and abridging, we have

$$\sin b \sin c \sin^2 \tfrac{1}{2} A = \sin \tfrac{1}{2} (a - b + c) \sin \tfrac{1}{2} (a + b - c) \quad (2)$$

Using the same notation as in art. 196, dividing (2) by $\sin b \sin c$, and extracting the square root, we have formula XXXII., viz.:

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}$$

227. From the same formula XXIX., by a process very similar to the preceding, but using formula II., art. 186, with the superior sign, and formula XVII., art. 192, we obtain formula XXXIII., viz.:

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}}.$$

228. By dividing the members of formula XXXII., by the corresponding members of formula XXXIII., we obtain formula XXXIV., viz.:

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}$$

This is the foundation of the rule in art. 176. It has only to be translated into logarithmic language before it be expressed in the words of the rule.

229. The similarity between the three preceding formulæ, and those contained in arts. 196, 197, 198, cannot fail to strike the attentive reader, and it is scarcely necessary to remark, that, by similar investigations, or symmetrically, similar formulæ may be found for the other angles B and C of a spherical triangle. By multiplying the members of formula XXXII., by the corresponding members of formula XXXIII., doubling each product, referring to formula XVI., art. 192, and abridging, we obtain formula XXXV., viz.:

$$\sin A = \frac{2\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}}{\sin b \sin c}.$$

230. By dividing the members of formula XXXV., by $\sin a$, we have formula XXXVI., viz.:

$$\frac{\sin A}{\sin a}=\frac{2\sqrt{\sin s \sin(s-a) \sin(s-b) \sin(s-c)}}{\sin a \sin b \sin c}$$

Now. as equations containing $\frac{\sin B}{\sin b}$ or $\frac{\sin C}{\sin c}$ in the first member, and having their second member exactly the same as the above, may be found by a similar process, or symmetrically; it follows, that

$$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b},\ \frac{\sin A}{\sin a}=\frac{\sin C}{\sin c},\ \text{and}\ \frac{\sin B}{\sin b}=\frac{\sin C}{\sin c};\ \text{or,}$$

$$\sin a : \sin A :: \sin b : \sin B :: \sin c : \sin C.$$

Consequently, *the sines of the sides of a spherical triangle, are to each other, as the sines of the opposite angles.* This is called the *Rule of Four Sines*, and is the foundation of the rules in arts. 181 and 182, which are only their translation into logarithmic language.

231. The analogies in the preceding rule may be expressed otherwise as follows:

$$\frac{\sin A}{\sin B}=\frac{\sin a}{\sin b},\ \frac{\sin A}{\sin C}=\frac{\sin a}{\sin c},\ \text{and}\ \frac{\sin B}{\sin C}=\frac{\sin b}{\sin c}.$$

In this form they will frequently be as useful as in the foregoing, as will be seen in the sequel.

232. By equating the values of $\cos c$ found from formulæ XXIX. and XXXI., freeing the equation of fractions, substituting $1-\sin^2 b$ for $\cos^2 b$, and dividing by $\sin b$, we obtain

$$\cos a \sin b = \cos A \sin c + \cos C \sin a \sin b;$$

now, dividing this equation by $\sin a$, and substituting $\frac{\sin C}{\sin A}$ for $\frac{\sin c}{\sin a}$, $\cot a$ for $\frac{\cos a}{\sin a}$ &c., we obtain formula XXXVII., viz.:

$$\cot a \sin b = \cot A \sin C + \cos C \cos b.$$

By reasoning in a similar manner, or symmetrically, concerning each separate pair of sides, we obtain the following additional formulæ, from XXXVIII. to XLII. inclusive, viz.:

$$\cot a \sin c = \cot A \sin B + \cos B \cos c$$

$$\cot b \sin a = \cot B \sin C + \cos C \cos a$$
$$\cot b \sin c = \cot B \sin A + \cos A \cos c$$
$$\cot c \sin a = \cot C \sin B + \cos B \cos a$$
$$\cot c \sin b = \cot C \sin A + \cos A \cos b$$

233. From the consideration of the *Polar Triangle* of any spherical triangle, it is easily seen that the measures of the angles of the latter, are the supplements of the sides of the former, and that the measures of the angles of the former, are the supplements of the sides of the latter. Consequently, if in any equation relating to a spherical triangle, we introduce $\pi - A$, $\pi - B$, $\pi - C$, instead of a, b, c, and $\pi - a$, $\pi - b$, $\pi - c$, instead of A, B, C, we shall have the resulting equation equally true with the original equation, and the same as would arise from a separate investigation. Making these substitutions in formulæ XXIX. to XXXI. inclusive, art. 215, and changing signs, we have the three following formulæ, XLIII. to XLV. inclusive, viz.:

$$\cos A = \cos a \sin B \sin C - \cos B \cos C$$
$$\cos B = \cos b \sin A \sin C - \cos A \cos C$$
$$\cos C = \cos c \sin A \sin B - \cos A \cos B$$

From these formulæ we can find the three sides of a spherical triangle, when the three angles are given; but they require the use of a table of natural sines.

234. Again, if in formulæ XXXII. to XXXV. inclusive, arts. 216, 217, 218, and 219, we make the same substitutions as we have mentioned in the preceding article, putting $S = \frac{1}{2}(A + B + C)$, and recollecting that $\sin \frac{1}{2}\left\{3\pi - (A + B + C)\right\} = -\cos S$, $\sin \frac{1}{2}\left\{\pi - (B + C - A)\right\} = \cos (S - A)$, &c., we shall have the four following formulæ, XLVI. to XLIX. inclusive, viz.:

$$\sin \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S - A)}{\sin B \sin C}}$$
$$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}}$$
$$\tan \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}}$$
$$\sin a = \frac{2\sqrt{-\cos S \cos (S - A) \cos (S - B) \cos (S - C)}}{\sin B \sin C}$$

Of these formulæ, the XLVIIIth. is the foundation of the rule in art. 177.

235. To determine formulæ for right-angled spherical triangles, we have only to suppose C a right angle in the formulæ art. 231, in formulæ XXXVII. XXXIX. XLI. XLII., art. 232, formulæ XLIII. XLIV. XLV., art. 233, and formula XXXI., art. 225; then, recollecting that $\cos C = 0$, $\sin C = 1$, and abridging the results, we obtain the five following double formulæ, L. to LIV. inclusive, viz.:

$$\sin a = \tan b \cot B = \sin c \sin A$$

$$\sin b = \tan a \cot A = \sin c \sin B$$

$$\cos c = \cot A \cot B = \cos a \cos b$$

$$\cos A = \tan b \cot c = \cos a \sin B$$

$$\cos B = \tan a \cot c = \cos b \sin A$$

By supplying the radius to render these formulæ homogeneous, we obtain from them, the three following theorems for the resolution of right-angled spherical triangles:

I. The rectangle of the radius and the sine of one of the legs, is equal to the rectangle of the cotangent of the adjacent oblique angle and the tangent of the other leg, or to the rectangle of the sines of the opposite angle and the hypotenuse.

II. The rectangle of the radius and the cosine of the hypotenuse, is equal to the rectangle of the cotangents of the oblique angles, or to the rectangle of the cosines of the legs.

III. The rectangle of the radius and the cosine of one of the oblique angles, is equal to the rectangle of the tangent of the adjacent leg and the cotangent of the hypotenuse, or to the rectangle of the cosine of the opposite leg and the sine of the other oblique angle.

236. The preceding rules are in effect the same as those commonly known by the title of *Naper's Rules for Circular Parts;* of these, Woodhouse says, "in the whole compass of mathematical science there cannot be found, perhaps, rules which more completely attain that which is the proper object of rules, namely, facility and brevity of computation." He then gives the rules and their description as follows:

"*Description of Circular Parts.*

The right angle is not considered; the complements of the other two angles, the complement of the hypotenuse, and the two sides, making in all five quantities, are called by Naper, *circular parts.* Any one of these circular parts may be called a *middle part* (M), and then, the two circular parts immediately adjacent to the right and left of M, the middle part, are called *adjacent parts;* the other two remaining circular parts, each separated from M, the middle part, by an adjacent part, are called *opposite parts*, or *opposite extremes:* thus, let the side a be M, the middle part; then,

complement of B, or $90^\circ - B$, and b, are the adjacent parts; and complement of c, or $90^\circ - c$, and $90^\circ - A$, are the opposite parts. If $90^\circ - A$, be M, the middle part; then $90^\circ - c$ and b, are the adjacent parts; and $90^\circ - B$ and a, are the opposite parts.

This necessary explanation being premised, we come to

Naper's Rules.

I. The rectangle of the sine of M, the middle part, and radius, is equal to the rectangle of the *tangents* of the *adjacent* parts.

II. The rectangle of the sine of M, the middle part, and radius, is equal to the rectangle of the *cosines* of the *opposite* parts.

To prove the truth of these rules, we have only to substitute, in formulæ L. to LIV. inclusive, which were established in the preceding article, the complements of A and B, the two acute angles, and the complement of c, the hypotenuse, instead of those angles, and that side themselves, and, recollecting that

$$\cos c = \sin (90^\circ - c),\ \sin c = \cos (90^\circ - c),$$
$$\cot c = \tan (90^\circ - c),$$
$$\cos A = \sin (90^\circ - A),\ \sin A = \cos (90^\circ - A),$$
$$\cot A = \tan (90^\circ - A),$$
$$\cos B = \sin (90^\circ - B),\ \sin B = \cos (90^\circ - B)$$
$$\cot B = \tan (90^\circ - B),$$

we have formulæ LV. to LIX. inclusive, viz.:

$$\sin a = \tan b \tan (90^\circ - B) = \cos (90^\circ - c) \cos (90^\circ - A)$$
$$\sin b = \tan a \tan (90^\circ - A) = \cos (90^\circ - c) \cos (90^\circ - B)$$
$$\sin (90^\circ - c) = \tan (90^\circ - A) \tan (90^\circ - B) = \cos a \cos b$$
$$\sin (90^\circ - A) = \tan b \tan (90^\circ - c) = \cos a \cos (90^\circ - B)$$
$$\sin (90^\circ - B) = \tan a \tan (90^\circ - c) = \cos b \cos (90^\circ - A)$$

These formulæ, thus demonstrated, are all concentrated in the preceding Rules called Naper's, and are the foundation of the rules delivered in art. 175.

237. We have only now to investigate the rules usually denominated *Naper's Analogies*. By formula I., art. 185, we have

$$\sin \tfrac{1}{2} (A + B) = \sin (\tfrac{1}{2} A + \tfrac{1}{2} B) = \sin \tfrac{1}{2} A \cos \tfrac{1}{2} B + \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B.$$

By substituting in this equation, the values of the four terms in the last member, obtained from formulæ XXXII. and XXXIII., arts. 226, 227, and abridging, we have

$$\sin \tfrac{1}{2} (A + B) = \frac{\sin (s - b) + \sin (s - a)}{\sin c} \sqrt{\frac{\sin s \sin (s - c)}{\sin a \sin c}}$$

Again, by referring the numerator of the first term of the second member of this equation, to formula VII., art. 189, the denominator to formula XVI., art. 192, and the surd term to art. 229, we have, by abridging the results, formula LX., viz.:

$$\sin \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2} c} \cos \tfrac{1}{2} C.$$

238. In the very same manner, we may very easily obtain the three following formulæ, LXI. LXII. LXIII., viz.:

$$\cos \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a + b)}{\cos \frac{1}{2} c} \sin \tfrac{1}{2} C$$

$$\sin \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2} c} \cos \tfrac{1}{2} C$$

$$\cos \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2} c} \sin \tfrac{1}{2} C$$

239. Now, by dividing the members of formula LX. by the corresponding members of LXI., we obtain formula LXIV., viz.:

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C$$

In the same manner, from formulæ LXII. and LXIII., we obtain formula LXV., viz.:

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \tfrac{1}{2} C.$$

240. By a method very similar to the preceding, from formulæ XLVI. and XLVII., art. 234, we should obtain the two following formulæ, LXVI. and LXVII., which, with the two just demonstrated, constitute what are called Naper's Analogies, viz.:

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \tfrac{1}{2} c$$

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \tfrac{1}{2} c$$

The four last formulæ, when translated into logarithmic language, will give the rules laid down in arts. 178, 179, and 180.

241. The different cases of oblique-angled spherical triangles may be solved by Naper's rules, by drawing a perpendicular to any side from the vertical angle; they will be more easily and directly solved, however, by the application of the three theorems in art. 235. Thus, in fig. 8, let AF be perpendicular to BC, and let θ represent the

segment BF of the base, and Θ the angle BAF, opposite to that segment; then $a - \theta$ will represent the other segment FC, and $A - \Theta$ the other angle FAC, opposite to it. Now, in the triangle BAF, we have, by theorem III., $\cos B = \tan \theta \cot c$; whence we obtain formula LXVIII., viz.:

$$\tan \theta = \cos B \tan c.$$

Again, in the same triangle, we have, by theorem II., $\cos c = \cot \Theta \cot B$; whence, we obtain formula LXIX., viz.:

$$\tan \Theta = \cot B \sec c.$$

242. Again, by finding expressions for the cosine of the perpendicular AF, in functions of the segments of the base and the adjacent sides, by theorem II., from both triangles BAF, FAC, and putting these expressions equal to each other, we have

$$\frac{\cos c}{\cos b} = \frac{\cos \theta}{\cos (a - \theta)}$$

whence, we readily obtain formula LXX., viz.:

$$\cos b = \cos c \cos (a - \theta) \sec \theta$$

From the preceding formula, we derive the following general theorem, relating to oblique-angled spherical triangles, viz.:

I. The cosines of the segments of the base intercepted between the perpendicular, and the extremities of the base, are to each other, as the cosines of the adjacent sides of the triangle.

243. Next, by finding expressions for the tangent of the perpendicular AF, in functions of the segments of the base and the adjacent angles, by theorem I., from both triangles as before, equating the values and reducing, we have

$$\frac{\sin \theta}{\sin (a - \theta)} = \frac{\cot B}{\cot C}$$

whence, we easily obtain formula LXXI., viz.:

$$\cot C = \cot B \sin (a - \theta) \operatorname{cosec} \theta$$

From the preceding formula, we obtain another general theorem relating to oblique-angled spherical triangles, viz.:

II. The sines of the segments of the base are proportional to the cotangents of the adjacent angles.

244. Thirdly, by finding expressions for the cosine of the perpendicular AF, in functions of the parts of the angle A, and the angles at the base, by theorem III., from both triangles as before; then, equating and reducing, we have

$$\frac{\sin \Theta}{\sin (A - \Theta)} = \frac{\cos B}{\cos C};$$

from which we easily deduce formula LXXII., viz.:

$$\cos C = \cos B \sin (A - \Theta) \operatorname{cosec} \Theta$$

Consequently, the next general theorem for oblique-angled spherical triangles is,

III. The sines of the angles contained by the sides, and the perpendicular, are to each other, as the cosines of the adjacent angles at the base.

245. Lastly, by similar methods, the following expressions may be obtained:

$$\frac{\cos \Theta}{\cos (A - \Theta)} = \frac{\cot c}{\cot b}, \quad \frac{\tan \theta}{\tan (a - \theta)} = \frac{\tan \Theta}{\tan (A - \Theta)}$$

whence, we have formulæ LXXIII. and LXXIV., viz.:

$$\cot b = \cot c \cos (A - \Theta) \sec \Theta$$

$$\tan (a - \theta) \tan \Theta = \tan (A - \Theta) \tan \theta.$$

Consequently, we have the two following general theorems for oblique-angled spherical triangles, viz.:

IV. The cosines of the angles contained by the sides and the perpendicular, are proportional to the cotangents of the sides.

V. The tangents of the same angles, are proportional to the tangents of the segments of the base.

246. In concluding the investigation of these rules, it will be useful to give the following general rule for rendering any trigonometrical formula *homogeneous*, whether it relate to plane or spherical triangles, or to the properties of Trigonometric functions in general; viz.: *Multiply each term by such a power of the radius, as will render it of the same dimensions as the term containing the greatest number of factors.* Thus, to render formula XXIX. homogeneous, we multiply the first term by R^2, and the third term by R; whence, we have

$$R^2 \cos a = \sin b \sin c \cos A + R \cos b \cos c;$$

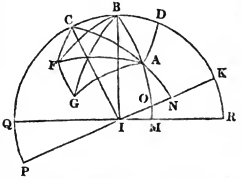
a formula, in which each term contains three factors; consequently, the terms are now all of the same dimensions, that is, they are homogeneous.

IV.—ASTRONOMICAL AND NAUTICAL PROBLEMS.

247. Given the right ascension and declination of a heavenly body, and the obliquity of the ecliptic; to find its latitude and longitude; and, conversely. Let A, fig. 9, be

the place of any heavenly body, B the pole of the equator QR, and C the pole of the ecliptic PK, and let PBR be the horary circle which joins the two poles; then, BC = KR, the obliquity of the ecliptic; AM the declination, and AB the co-declination or polar distance; AN the latitude, and AC the co-latitude; IM the right ascension, and the angle ABC = 90° + the right ascension; IN the longitude, and the angle ACB = 90° — the longitude. Denoting BD, the arc intercepted between D, the foot of the perpendicular AD, and B the vertex of the angle ABC, by θ, we have, by formula LXVIII, $\tan\theta = -\cos B \tan c$; whence we obtain formula LXXV, viz.:

fig. 9.



$$\tan\theta = \sin \text{A.R.} \cot \text{dec.}$$

Again, by formula LXX., we have $\cos b = \cos c \cos (a + \theta) \sec\theta$; whence we have formula LXXVI., viz.:

$$\sin \text{lat.} = \sin \text{dec.} \cos (\text{obl.} + \theta) \sec\theta.$$

Next, by theorem III., art. 235, we have formula LXXVII., viz.:

$$\sin \text{lon.} = \tan \text{lat.} \tan (\text{obl.} + \theta)$$

These three formulæ are true for every possible position of the heavenly body A, by observing that when the latitude and declination are south, these quantities are to be considered as negative, and that the longitude and right ascension are always of the same affection.

248. Another expression for the longitude might have been obtained from formula LXXI., viz.:

$$\tan \text{lon.} = \tan \text{A.R.} \sin (\text{obl.} + \theta) \operatorname{cosec}\theta.$$

The angle BAC, which is called the *angle of position*, is easily found by the rule of Four Sines. Thus, denoting the angle of position by P, we have the following formula, LXXVIII., viz.:

$$\sin P = \cos \text{A.R.} \sin \text{obl.} \sec \text{lat.} = \cos \text{lon.} \sin \text{obl.} \sec \text{dec.}$$

249. Conversely, denoting CD, the arc intercepted between D the foot of the perpendicular, and C the vertex of the angle ACB, by ϕ, we should, by processes similar to the preceding, obtain the following formulæ from LXXIX. to LXXXII., inclusive, viz.:

$\tan \phi = \sin \text{lon.} \cot \text{lat.}$
$\sin \text{dec.} = \sin \text{lat.} \cos (\phi - \text{obl.}) \sec \phi$
$\sin \text{A.R.} = \tan \text{dec.} \tan (\phi - \text{obl.})$
$\tan \text{A.R.} = \tan \text{lon.} \sin (\phi - \text{obl.}) \operatorname{cosec} \phi.$

Ex. 1. In the *Nautical Almanac* for 1838, the mean obliquity of the ecliptic has been taken $= 23° 27' 37''.43$, on January 1st; and in page V. of the month, at noon of the same day, the moon's right ascension is 23 hours, 18 min., 4.64 sec., and her declination is $7° 20' 8''.5$ S. Required her latitude and longitude?

Here, 23 hours, 18 min., 4.64 sec., the A.R. in time $= 349° 31' 9''.6$; whence,

$$
\begin{aligned}
\log \sin 349° \; 31' &= 9.259951 - \\
\log \cot - 7 \;\; 20 &= 0.890441 - \\
\log \tan \; 54 \;\; 44 &= \overline{0.150392} \\
\log \sin - 7 \;\; 20 &= 9.105992 - \\
\log \cos \; 78 \;\; 11 &= 9.311289 \\
\log \sec \; 54 \;\; 44 &= 0.238536 \\
\log \sin - \; 2 \;\; 36 &= \overline{8.655817} - \\
\log \tan - \; 2 \;\; 36 &= 8.657149 - \\
\log \tan \; 78 \;\; 11 &= 0.679408 \\
\log \sin 347 \;\; 28 &= \overline{9.336557} - \\
\text{or, } \log \tan 349 \;\; 31 &= 9.267261 - \\
\log \sin \; 78 \;\; 11 &= 9.990697 \\
\log \operatorname{cosec} \; 54 \;\; 44 &= 0.088058 \\
\log \sin 347 \;\; 30 &= \overline{9.346016} -
\end{aligned}
$$

Consequently, by the above calculation, which is taken to the nearest minute in the tables, the moon's latitude is $2° 36'$ S., and her longitude $347° 30'$. In the *Nautical Almanac*, page IV. of the month, the latitude is $2° 35' 46''.6$ S., and the longitude $347° 29' 35''.1$. The second method of finding the longitude is evidently the best in practice.

Ex. 2. On the 1st January, 1830, the latitude of the star Aldebaran, or the Bull's Eye, was $5° 29'$ S., and the longitude $67° 25'$. Required its right ascension and declination, and the angle of position?

$$
\begin{aligned}
\text{Here, } \log \sin 67° \; 25' &= 9.965353 \\
\log \cot - 5 \;\; 29 &= 1.017749 - \\
\log \tan 95 \;\; 56 &= \overline{0.983102} - \\
\log \sin - 5 \;\; 29 &= 8.980259 - \\
\log \cos 72 \;\; 28 &= 9.478942 \\
\log \sec 95 \;\; 56 &= 0.985600 - \\
\log \sin 16 \;\; 10 &= \overline{9.444801} \\
\log \tan 16 \;\; 10 &= 9.462242 \\
\log \tan 72 \;\; 28 &= 0.500397 \\
\log \sin 66 \;\; 34 &= \overline{9.962639}
\end{aligned}
$$

$$\begin{aligned}
\text{or, log tan } 67\ \ 25 &= 0.380992\\
\text{log sin } 72\ \ 28 &= 9.979340\\
\text{log cosec } 95\ \ 56 &= \underline{0.002333}\\
\text{log tan } 66\ \ 33 &= \underline{0.362665}
\end{aligned}$$

Consequently, the right ascension of Aldebaran is 66° 33′, or about 4 hours, 26 min., and the declination 16° 10′ N. In this case, also, the second method of finding the right ascension is the best in practice.

$$\begin{aligned}
\text{Again, log cos } 66^\circ\ 33' &= 9.599827\\
\text{log sin } 23\ \ 28 &= 9.600118\\
\text{log sec} - 5\ \ 29 &= \underline{0.001992}\\
\text{log sin } 9\ \ 9\tfrac{1}{2} &= \underline{9.201937}
\end{aligned}$$

whence the angle of position of Aldebaran is about 9° 9½′.

250. When the preceding problems are applied to the sun, the formulæ are much simplified, in consequence of the latitude being reduced to zero; the solution by means of an auxiliary arc, becomes then unnecessary; for, referring to the right-angled spherical triangle OIM, fig. 9, where IM represents the right ascension of the sun, OM his declination, IO his longitude, and the angle MIO the obliquity of the ecliptic, we have, by the application of *Naper's Rules*, art. 236, and the principles of Trigonometry, the following formulæ, LXXXIII. to XC., inclusive, viz.:

$$\begin{aligned}
\tan \text{A.R.} &= \cos \text{obl.} \tan \text{lon.}\\
\sin \text{A.R.} &= \cot \text{obl.} \tan \text{dec.}\\
\tan \text{dec.} &= \sin \text{A.R.} \tan \text{obl.}\\
\sin \text{dec.} &= \sin \text{lon.} \sin \text{obl.}\\
\cos \text{lon.} &= \cos \text{A.R.} \cos \text{dec.}\\
\sin \text{lon.} &= \sin \text{dec.} \operatorname{cosec} \text{obl.}\\
\cos \text{obl.} &= \tan \text{A.R.} \cot \text{lon.}\\
\tan \text{obl.} &= \tan \text{dec.} \operatorname{cosec} \text{A.R.}
\end{aligned}$$

Ex. 1. On the 1st of January, 1838, per *Nautical Almanac*, the sun's longitude was 280° 42′ 36″.4. Required his right ascension and declination?

$$\begin{aligned}
\text{log. cos. } 23^\circ\ 28' &= 9.962508\\
\text{log. tan } 280\ \ 43 &= \underline{0.722957} -\\
\text{log. tan } 281\ \ 39 &= \underline{0.685465} -\\[4pt]
\text{log sin } 281^\circ\ 39' &= 9.990960 -\\
\text{log tan } 23\ \ 28 &= \underline{9.637611}\\
\text{log tan} - 23\ \ \ 2 &= \underline{9.628571} -
\end{aligned}$$

Whence, the sun's right ascension is 281° 39′, and his declination 23° 2′ S. In the *Nautical Almanac*, the right as-

cension for the same day, is 18 hours, 46 min., 36.73 sec. = 281° 39′ 0″.3, and the declination 23° 1′ 48″.6 S.

Ex. 2. Given the sun's right ascension and declination, as above: required his longitude, and the obliquity of the ecliptic?

$$\begin{aligned}
\log\cos 281^\circ\ 39' &= 9.305207 - \\
\log\cos -23\quad 2 &= 9.963919 \\
\hline
\log\cos 280\quad 43 &= 9.269126 - \\[6pt]
\log\tan -23\quad 2 &= 9.628554 - \\
\log \text{cosec}\, 281\quad 39 &= 0.009040 - \\
\hline
\log\tan\ 23\quad 28 &= 9.637594 +
\end{aligned}$$

Whence, the sun's longitude is 280° 43′, and the obliquity of the ecliptic 23° 28′, both taken to the nearest minute.

251. Given the true altitude of a heavenly body, its declination, and the latitude of the place, to find the hour-angle, or distance of the body from the meridian. In fig. 9, let PK represent the horizon, I the place of the observer, C his zenith, B the pole as before, and A the place of a heavenly body. Then, AN represents the altitude, and AC the zenith distance; AM the declination, and AB the polar distance; BK the latitude, and BC, the co-latitude; also, in the triangle ABC, the angle ABC represents the hour angle or meridian distance. Whence, $\frac{1}{2}$ (180 + *pol. dist.* — *alt.* — *lat.*) = half the sum of the three sides AB, BC, CA, and subtracting from this half sum, each of the sides, the polar distance and the co-latitude successively, we have the two remainders, $\frac{1}{2}$ (180 — *pol. dist.* — *alt.* — *lat.*) = 90 — $\frac{1}{2}$ (*pol. dist.* + *alt.* + *lat.*), and $\frac{1}{2}$ (*pol. dist.* + *lat.* — *alt.*) = $\frac{1}{2}$ (*pol. dist.* + *lat.* + *alt.*) — *alt.* Now, denoting the angle ABC by D, the former of these remainders by 90 — H, and the latter by R, and observing that $\sin (90 - \text{H}) = \cos \text{H}$, $\dfrac{1}{\sin colat.} = sec\ lat.$, $\dfrac{1}{\sin pol.\ dist.} =$ cosec *pol. dist.*, we have, by squaring formula XXXII., art. 226, and making the proper substitutions of these values,

$$\sin^2 \tfrac{1}{2}\,\text{D} = \cos \text{H} \sin \text{R} \sec \text{lat.} \operatorname{cosec} \text{pol. dist.}$$

Now, by putting this equation into logarithms, we have formula XCI., viz.:

$$2 \log \sin \tfrac{1}{2}\,\text{D} = \log\cos \text{H} + \log\sin \text{R} + \log\sec lat. + \log \operatorname{cosec} pol.\ dist. - 20.$$

To facilitate the computation of the meridian distances by this formula, tables are constructed and inserted in books of Navigation, containing twice the log. sines of half the

meridian distances or hour angles, within certain limits proper for such observations; so, that the answer may be obtained at once, without the trouble of dividing the sum of these four logarithms by 2, finding the angle corresponding to the log. sine thus obtained, doubling it, and converting the degrees, &c., into time.

252. The rules for finding the hour angle, commonly inserted in Treatises on Navigation, is as follows, which is merely a translation of the foregoing formula: "Add together the body's true altitude, the polar distance, and the latitude of the place of observation; take the difference between half their sum and the body's true altitude, and note the remainder; then, add together the cosecant of the polar distance, the secant of the latitude, the cosine of the half sum, and the sine of the remainder. The sum of these four logarithms, rejecting tens in the index, will be the log. (in such a table as we have mentioned in the preceding art.) answering to the meridian distance or hour angle required." If the heavenly body be the sun, it is evident that the meridian distance will be the apparent time from the nearest noon. If any other body be observed to find the apparent time, the meridian distance, or horary angle, must be added to the difference of the right ascension of the body, and the right ascension of the sun, when the body is to the west of the meridian; and, it must be subtracted from that difference, when that body is to the east of the meridian. It must be observed, that in finding the difference of the right ascensions, the right ascension of the body is always to be subtracted from that of the sun, increased by 24 hours, if necessary. This problem is highly useful in finding the longitude of a ship at sea, either by chronometers, or by lunar observations.

Ex. On the 16th of August, 1835, in lat. 36° 31′ N., and long. by account, 152° E., at 4 hours, 42 min., P.M., per watch, the altitude of the sun's lower limb was 23° 50′ 24″, the height of the observer's eye being 20 feet. Required the apparent and mean time at ship, and the error of the watch?

Here, by correcting the sun's altitude for dip, refraction, parallax, and semidiameter, we find the true altitude 23° 59′ 56″; by reducing the apparent time at the ship per watch to Greenwich mean time, we find it to be 18 hours, 38 min., P.M., on August 15th; by correcting the sun's declination on 15th August, as given in the *Nautical Almanac*, for 18 hours, 38 min., we find the true declination 13° 58′ 45″ N., and, consequently, the polar distance 76° 1′ 15″; and by correcting the equation of time on August 15th, for Greenwich mean time, we find the true equation + 4 min.,

11 sec. Whence, by the rule, taking the data to the nearest minute, we have

Sun's true alt.	24° 0′	
Sun's pol. dist.	76 1	log cosec 10.013064
Lat. N.	36 31	log sec 10.094915
Sum	136 32	
Half sum	68 16	log cosine 9.568539
Remainder	44 16	log sine 9.843855
App. ship time	4h. 41m. 11s.	log 9.520373
Eq. of time	+ 4 11	
Mean ship time	4 45 22 P.M.	
Time per watch	4 42 6 P.M.	
Error of watch	0 3 16	

If we substract 20 from the sum of these four logs. we have 19.520373, which halved, gives 9.760187; now, this is the log. sine of 35° 8′ 52″, which doubled and converted into time, gives 4 hours, 41 min., 11 sec., P.M., the apparent time at the ship; and applying the equation of time, we find the mean time at ship 4 hours, 52 min., 22 sec., P.M., and consequently, the error of the watch is 3 min., 16 sec.

253. Given the right ascensions and declinations of two heavenly bodies, to find the distance between them. Let F and G, fig. 9, be the positions of two heavenly bodies; and B the pole of the equator; then, FB, GB, represent their codeclinations; the angle FBG, the difference of their right ascensions; and, FG, the distance between them. Denoting their right ascensions by a, a'; their declinations by d, d', and their distance by D; we have, in the triangle BFG, by formulæ LXVIII. and LXX.,

$$\tan\,\theta = \tan\,(90° - d)\cos\,(a - a')$$
$$\cos D = \cos\,(90° - d)\cos\,(90° - d - \theta)\sec\,\theta$$

Whence, we derive the following formulæ XCII. and XCIII., viz.

$$\tan\,\theta = \cot d\cos\,(a - a')$$
$$\cos D = \sin d\sin\,(d' + \theta)\sec\,\theta$$

in which, when the declinations are south, and when a is less than a', the arcs d, d', and $(a - a')$ must be considered as negative.

Ex. In the *Nautical Almanac* for 1838, on the 1st January at mean noon, the sun's right ascension is 18 hours, 46 min., 36.02 sec., and declination 23° 1′ 49″.4 S.; and, the moon's right ascension is 23 hours, 18 min., 4.64 sec., and declination 7° 20′ 8″.5 S.; required the distance between the sun and moon, at that instant.

Here, $(a - a') = 4$ hours, 31 min., 28.62 sec., $= - 67° 52' 9''.3$, $d = - 23° 2'$, and $d' = - 7° 20''$; whence

$$\begin{aligned}
\log\cot -\ 23^\circ\ 2' &= 10.371446- \\
\log\cos -\ 67\ \ 52\ &= \ \ 9.576069 \\
\log\tan\ 138\ \ 27\ &= \ \ \overline{9.947515}- \\[4pt]
\log\sin -\ 23\ \ \ 2\ &= \ \ 9.592473- \\
\log\sin\ 131\ \ \ 7\ &= \ \ 9.877010 \\
\log\sec\ 138\ \ 27\ &= 10.125879- \\
\log\cos\ 66\ \ 49\ &= \ \ \overline{9.595362}
\end{aligned}$$

Thus, the required distance is 66° 49′ to the nearest minute; in the *N. Almanac*, p. XIII. of the month, the distance is 66° 48′ 30″.

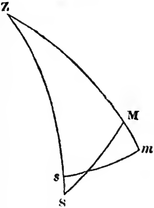
254. If, instead of the right ascensions and declinations, of the two heavenly bodies, their latitudes and longitudes were given, to find the distance between them; the formulæ of solution would be the same as above, with this difference only, that $a - a'$ would be the difference of their longitudes, and d, d', would be their colatitudes.

255. Connected with the preceding problem, is the *method of finding the longitude at sea by lunar observations.* The refraction and parallax of the heavenly bodies occasion observers at the surface of the earth, to refer them to places in the celestial sphere, different from those to which they would be referred at the centre of the earth, or of the celestial sphere. The apparent place of the sun's limb or a star is always higher than its true place; and the apparent place of the moon's limb is always lower than its true place, the moon's refraction, in consequence of its vicinity to the earth, being always less than its parallax. Now, as the *Nautical Almanac* contains the *true* distances between the moon and the sun and certain stars and planets near the ecliptic, calculated for every 3 hours in the day during the whole year, at Greenwich mean time; if, the *apparent* distance between the moon and any of these bodies be observed at any other place of the globe, and then by calculation, be cleared of the effects of parallax and refraction, mean time at that place being known or calculated from the observed height of one of the heavenly bodies at that instant, —then, the difference of the meridians of the two places in time, and consequently, the longitude, becomes known. It is plain that for the complete solution of this problem, the true distances of the moon and the heavenly bodies alluded to, would require to be known at every instant; but intervals of time less than 3 hours, can be determined with sufficient accuracy for practical purposes, by proportion, or by interpolation.

256. To clear the apparent distance between the moon and a heavenly body, from the effects of parallax and refraction; or, in other words, to find the true distance from the apparent distance, the true and apparent altitudes of the centres

of those bodies must be known. In fig. 10, let m represent the apparent, and M the true place of the moon; s the apparent, and S the true place of the sun or a star; and let Z be the zenith of the observer. Then mZ, sZ will represent the apparent zenith distances; MZ, SZ, the true zenith distances; mM, the correction of the moon's altitude; sS, the correction of the sun's or star's altitude; ms, the apparent distance; and, MS the true distance. Now, putting $ms = \Delta$, $MS = D$, $mZ = 90° - h$, $sZ = 90° - h'$, $MZ = 90° - H$, and $SZ = 90° - H'$; we have by transposition and division from formula XXIX.,

fig. 10.



$$\cos MZS = \frac{\cos\Delta - \sin h \sin h'}{\cos h \cos h'} = \frac{\cos D - \sin H \sin H'}{\cos H \cos H'}$$

By adding unity to each of the fractions, in this double equation, reducing the results to a common denominator, and abridging by formula II. art. 186, we have

$$\frac{\cos\Delta + \cos(h + h')}{\cos h \cos h'} = \frac{\cos D + \cos(H + H')}{\cos H \cos H'} \quad (1)$$

Again putting $S = \frac{1}{2}(h + h' + \Delta)$, and applying formula IX. art. 189, we have

$$\cos\Delta + \cos(h + h') = 2\cos\tfrac{1}{2}(h + h' + \Delta)\cos\tfrac{1}{2}(h + h' - \Delta) = 2\cos S\cos(S - \Delta) \quad (2)$$

Now, by formulæ XVII. and XVIII. art 192, we have

$$1 + \cos(H + H') = 2\cos^2\tfrac{1}{2}(H + H')$$
$$1 - \cos D = 2\sin^2\tfrac{1}{2}D$$

Whence, by subtraction, we obtain

$$\cos(H+H') + \cos D = 2\cos^2\tfrac{1}{2}(H+H') - 2\sin^2\tfrac{1}{2}D \quad (3)$$

By substituting in equation (1), the values of the numerators of the fractions of its members, obtained in equations (2) and (3), transposing and abridging, we have

$$\sin^2\tfrac{1}{2}D = \cos^2\tfrac{1}{2}(H + H') - \frac{\cos H\cos H'}{\cos h\cos h'}\cos S\cos(S-\Delta) \quad (4)$$

Lastly, to adapt this equation for calculation by logarithms, put

$$\sin\phi = -\frac{\sqrt{\frac{\cos H \cos H'}{\cos h \cos h'}}\cos S \cos (S - \Delta)}{\cos \frac{1}{2}(H + H')} \quad (5)$$

Then, by substitution and reduction, we have

$$\sin \tfrac{1}{2} D = \cos \tfrac{1}{2}(H + H') \cos \phi \quad (6)$$

Thus, S and ϕ being known, we find D.

257. The preceding formulæ may be embodied in the words of the following rule, which will be found useful when a variety of tables is not at hand, as it requires only the application of Table III. of the "*Practical Mathematician's Pocket Guide*," the necessary preparatory tables being given at the end of this volume. It must be premised that to find the apparent altitudes and distance of the heavenly bodies from their observed altitudes and distance, corrections must be applied in the usual way, for index error, semidiameter and depression of the horizon; and, that to find the true altitudes, corrections must be made for refraction and parallax.

Rule, To find the true Distance. Add together the apparent altitudes and the apparent distance, and take half their sum; take the difference between this half sum and the apparent distance. Then, add together the logarithmic cosines of the half sum, the difference, and the true altitudes; from their sum, subtract the sum of the logarithmic cosines of the apparent altitudes, and divide the remainder by 2. From this quotient, increased by 10, subtract the logarithmic cosine of half the sum of the true altitudes, and find the arc of which the remainder is the logarithmic sine. Add the logarithmic cosine of this arc, to the logarithmic cosine of half the sum of the true altitudes, already found, and the sum, minus 10, is the logarithmic sine of *half* the *true Distance.*

To this rule, it may be added that the mean time of the observation, at Greenwich is found thus: Take from the *Nautical Almanac*, the two distances between which the true distance falls, and find their difference; find also the difference between the true distance and the first of these distances; then, say, as the first difference is to the second, so is 3 hours to the proportional time, to be added to the time of the first distance given in the *Nautical Almanac;* the sum will be the mean time required. Consequently, the difference between the mean time at Greenwich and the mean time at the ship, found by art. 252, will be the longitude in time; *East*, if the mean time at the ship be greater than the mean time at Greenwich, reckoned from the same noon; but *West*, if the mean time at the ship, be less than the mean time at Greenwich.

Ex. June 17th, 1835, in lat. 29° 8′ N., and long. by account 147° 15′ W., about 8 hours, 32 min. A.M., the distance between the nearest limbs of the sun and moon was observed to be 92° 50′ 45″, the altitude of the moon's upper limb being 39° 1′ 20″, and the sun's lower limb 42° 36′ 20″; the height of the observer's eye 18 feet, and the instruments free of index error; required the longitude of the place of observation.

Here, the estimated mean time at Greenwich, ascertained from the longitude by account and the supposed time at the ship, is 6 hours, 21 min., P.M., June 17th. The moon's semidiameter and horizontal parallax, taken from the *Nautical Almanac*, for 1835, to the nearest noon, and corrected for Greenwich time, (and the semidiameter for augmentation) are 15′ 17″ and 55′ 28″. The apparent altitudes of the sun and moon, ascertained by correcting the observed altitudes for semidiameter and dip, are 42° 48′ 2″ and 38° 41′ 59″; while the true altitudes ascertained from these, by correction for refraction and parallax, are 42° 47′ 7″ and 39° 24′ 5″. The apparent distance of the centres of the sun and moon, found by adding their corrected semidiameters to the observed distance, is 93° 21′ 48″. Whence

App. dist.	93° 21′ 48″	
Sun's app. alt.	42 48 2	log cos = 9.865533
Moon's app. alt.	38 41 59	log cos = 9.892336
Sum	174 51 49	19.757869 −
Half sum	87 25 55	log cos = 8.651336
Difference	5 55 53	log cos = 9.997669
Sun's true alt.	42 47 7	log cos = 9.865639
Moon's true alt.	39 24 5	log cos = 9.888021
Sum	82 11 12	38.402665 +
Half sum	41 5 36	2)18.644796
		9.322398
9.877164 = log cos	41 5 36	log cos = 9.877164
9.982435 = log cos	16 11 11	log sin = 9.445234
9.859599 = log sin	46 21 59	
	2	
True distance	92 43 58	

This distance falls between 92° 53′ 44″, the distance at 6 hours, in the *Nautical Almanac*, and 91° 28′ 10″ the distance at 9 hours; whence, taking the differences, we have, by proportion, 0 hours, 20 min., 33 sec., for the proportional time due to the difference between the true distance, and the distance at 6 hours; consequently, the mean time of the observation at Greenwich is 6 hours, 20 min., 33 sec., June 17th, or 30 hours, 20 min., 33 sec, June 16th. Now,

by the rule given in art. 252, the mean time of the observation at the place where the apparent distance was taken, is 20 hours, 29 min., 41 sec. Consequently, the difference is the longitude in time, which, converted into degrees, makes the longitude 147° 43′ West.

The preceding rule, for finding the true distance, has some advantages over other methods given in treatises on Navigation; it requires reference to one table only, namely, the table of the logarithmic cosines and sines; all the logarithms to be taken out are for cosines, and all the arcs to be taken out are for sines; the proportional parts of the logarithms for the seconds of the arcs are all to be subtracted for cosines, and the seconds corresponding to the proportional parts are to be added for sines.

258. Given the distances of a heavenly body from two others whose positions are known, to find its right ascension and declination. Let G, fig. 9, be the heavenly body whose distances from two others, A and F, are given. In the triangle FBA, find AF, and the angle BFA. Then, in the triangle AFG, the three sides will be known; from these, find the angle AFG, and the angle BFG will be the sum or difference of the angles BFA, AFG, according as the points B and G are on the opposite sides, or the same side of AF. Again, in the triangle BFG, the sides BF, FG, and the contained angle BFG, are known; from which, find BG the polar distance, and the angle FBG, the difference of the right ascensions of the bodies F and G; from these the right ascension and declination of the body G may easily be found. Were the latitudes and longitudes of the bodies A and F given, the latitude and longitude of the body G, would be found in the very same way as the right ascension and declination. The following exercise on this problem may be solved by the learner.

Ex. On the 1st of January, 1839, at noon, the distances of Mars and Jupiter from the Moon, were 56° 29′ 41″ and 78° 19′ 40″ respectively; the right ascensions of these bodies at the same instant, being 11 hours, 47 min., 16.93 sec., and 13 hours, 5 min., 9.24 sec., respectively; and their declinations 4° 24′ 20″.1 N., and 5° 31′ 42″.1 S., respectively. Required the Moon's right ascension and declination. *Ans.* 8 hours, 7 min., 21.52 sec., and 24° 39′ 46″.3 N.

259. In concluding these problems, it may be useful to propose the investigation of the following formulæ, respecting the rising and setting of the heavenly bodies, their altitude, amplitude, azimuth, semidiurnal arc, &c., for the exercise of the learner, as well as for the purpose of reference when occasion may require. Thus, let P denote the semidiurnal arc, or measure of the time between noon and the hour of rising or setting; P′ the measure of the time

between noon and the hour of being due east or west, and a' the altitude at the same time; a the altitude at 6 o'clock; Z the azimuth from the north at rising or setting; Z' the azimuth from the north at 6 o'clock; and m the amplitude. Then, we have the formulæ from XCIV. to C. inclusive, viz.:

$$\cos P = -\tan \text{lat.} \tan \text{dec.}$$
$$\sin \text{dec.} = \cos \text{lat.} \cos Z$$
$$\sin \text{dec.} = \cos \text{lat.} \sin m$$
$$\sin a = \sin \text{lat.} \sin \text{dec.}$$
$$\cos \text{lat.} = \cot \text{dec.} \cot Z'$$
$$\sin \text{dec.} = \sin \text{lat.} \sin a'$$
$$\cos P' = \cot \text{lat.} \tan \text{dec.}$$

If in these formulæ, the latitude, declination, or amplitude, be south, the sine, tangent, and cotangent, must be considered as negative; and, if $\sin a$ or $\sin a'$ be found negative, a and a' would then denote *depression* below the horizon, instead of *altitude* or *elevation* above the horizon. From formula XCIV., it is evident, that a body will not reach the horizon if the latitude be greater than the co-declination, and from formula C., that a body cannot be on the prime vertical if its declination be greater than the latitude. In all these formulæ, no allowance is made for parallax, refraction, &c.; so that in their practical application, care must be observed in making such allowances according to the data of each particular problem.

V.—SURVEYING.

260. Besides the rules for casting up the superficial contents or areas of different geometrical figures, given in the articles on Mensuration, we may add the following general rule, leaving its investigation to the ingenuity of the learner.

Rule. To find the area of any field whose boundaries are straight lines.

1. Place in parallel columns, the courses and distances of the outlines of the field whose area is to be calculated, in the order in which they occur in measuring them, while traversing around it.

2. Put the differences of latitude and the departures of each line, opposite the course and distance of that line, in two different columns, parallel to the former.

3. Distinguish the northings and southings among the several differences of latitude from each other, by the signs + and —; and do the same with the eastings and westings among the departures. Or, call the first difference of lati-

tude, and all of the same name with it, *affirmative*, and all of a contrary name, *negative;* and, do the same with the departures.

4. Take the first departure and place it in a new column, parallel to the former, as the first of a column of multipliers. For the second multiplier, add together the first multiplier, the first departure, and the second departure; and generally, to find any multiplier, add together these three quantities, namely, the last preceding multiplier, the departure belonging to it, and the next succeeding departure. Observe that the number of multipliers must be the same as the number of the outlines of the field, a multiplier to each outline; and, that in adding + and — quantities, the rules of algebra must be strictly attended to.

5. Multiply each multiplier by the difference of latitude standing against it, and put the products in a new column, parallel to the former; take half the sum of these products, and the result will be the area of the field.

Note. If the multipliers be correctly obtained, the last multiplier will be equal to the last departure taken with a contrary sign.

Ex. In a field of four sides, the courses and distances of its four corners in order, were as follows: S. 60° E. 9 chains, S. 35° W. 8 chains, N. 70° W. 14 chains, and N. 57° 51′ E. 11.76 chains; required its area.

Courses.	Dist.	Lat.	Dep.	Mult.	Doubl.areas.
S. 60° E.	9 00	4.50	7.79	7.79	35.0550
S. 35° W.	8.00	6.55	— 4.59	10.99	71.9845
N. 70° W.	14.00	—4.79	—13.16	—6.76	32.3804
N. 57° 51′ E.	11.76	—6.26	9.96	—9.96	62.3496

Sum 201.7695

Ans. Area, Acres 10.08847

SECT. IV.— ASTRONOMICAL AND NAUTICAL TABLES.

The following tables, chiefly astronomical and nautical, are an indispensable addenda to those given in the "*Practical Mathematician's Pocket Guide,*" being necessary in the solution of numerous problems in Mensuration, Levelling, Astronomy, and Navigation. The titles and arguments of these tables, which are numbered according to the letters of the alphabet, are so plain, that there appeared to be not the slightest need for prefixing precepts for their use and application.

TABLE VII.—DIGITAL MULTIPLES

OF THE COMMON MODULUS AND ITS RECIPROCAL;

Their values to 30 decimal places, being as follows:

M = 0.434294481903251827651128918917

1 : M = 2.302585092994045684017991454684

N.	Common Modulus.	Reciprocal.	N.
1	0.434294481903252	2.302585092994046	1
2	0.868588963806504	4.605170185988091	2
3	1.302883445709755	6.907755278982137	3
4	1.737177927613007	9.210340371976183	4
5	2.171472409516259	11.512925464970228	5
6	2.605766891419511	13.815510557964274	6
7	3.040061373322763	16.118095650958320	7
8	3.474355855226015	18.420680743952365	8
9	3.908650337129266	20.723265836946411	9

TABLE VIII.

DIFFERENCE OF LATITUDE AND DEPARTURE,

For every Quarter Point of the Compass.

Cou	Dist. 1.		Dist. 2.		Dist. 3.		Cou
Pts.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Pts.
0¼	.99880	.04907	1.9976	0.0981	2.9964	0.1472	7¾
0½	.99518	.09802	1.9904	0.1960	2.9856	0.2941	7½
0¾	.98918	.14674	1.9784	0.2935	2.9675	0.4402	7¼
1	.98079	.19509	1.9616	0.3902	2.9424	0.5853	7
1¼	.97003	.24298	1.9401	0.4860	2.9101	0.7289	6¾
1½	.95694	.29028	1.9139	0.5806	2.8708	0.8709	6½
1¾	.94154	.33689	1.8831	0.6738	2.8246	1.0107	6¼
2	.92388	.38268	1.8478	0.7654	2.7716	1.1481	6
2¼	.90399	.42756	1.8080	0.8551	2.7120	1.2827	5¾
2½	.88192	.47140	1.7638	0.9428	2.6458	1.4142	5½
2¾	.85773	.51410	1.7155	1.0282	2.5732	1.5423	5¼
3	.83147	.55557	1.6629	1.1111	2.4944	1.6667	5
3¼	.80321	.59570	1.6064	1.1914	2.4096	1.7871	4¾
3½	.77301	.63439	1.5460	1.2688	2.3190	1.9032	4½
3¾	.74095	.67156	1.4819	1.3431	2.2229	2.0147	4¼
4	.70711	.70711	1.4142	1.4142	2.1213	2.1213	4
Pts.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Pts.

Cou	Dist. 4.		Dist. 5.		Dist. 6.		Cou
Pts.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Pts
0¼	3.9952	0.1963	4.9940	0.2453	5.9928	0.2944	7¾
0½	3.9807	0.3921	4.9759	0.4901	5.9711	0.5881	7½
0¾	3.9567	0.5869	4.9459	0.7337	5.9351	0.8804	7¼
1	3.9231	0.7804	4.9039	0.9755	5.8847	1.1705	7
1¼	3.8801	0.9719	4.8502	1.2149	5 8202	1.4579	6¾
1½	3.8278	1.1611	4.7847	1.4514	5.7416	1.7417	6½
1¾	3.7662	1.3476	4.7077	1.6844	5.6493	2.0213	6¼
2	3 6955	1.5307	4.6194	1.9134	5.5433	2.2961	6
2¼	3.6160	1.7102	4.5199	2.1378	5.4239	2.5653	5¾
2½	3.5277	1.8856	4.4096	2.3570	5.2915	2.8284	5½
2¾	3.4309	2.0564	4.2886	2.5705	5.1464	3.0846	5¼
3	3.3259	2.2223	4 1573	2.7779	4.9888	3.3334	5
3¼	3.2128	2.3828	4.0160	2.9875	4.8192	3.5742	4¾
3½	3.0920	2.5376	3 8650	3.1720	4.6381	3.8064	4½
3¾	2.9638	2.6862	3 7048	3.3578	4.4457	4.0294	4¼
4	2.8284	2.8284	3.5355	3.5355	4.2426	4.2426	4
Pts.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Pts

Cou	Dist. 7.		Dist. 8.		Dist. 9.		Cou
Pts.	Lat.	Dep	Lat.	Dep.	Lat.	Dep.	Pts.
0¼	6 9916	0 3435	7.9904	0.3925	8.9892	0.4416	7¾
0½	6.9663	0.6861	7.9615	0.7841	8.9567	0.8822	7½
0¾	6.9242	1.0271	7.9134	1.1738	8.9026	1.3206	7¼
1	6.8655	1.3656	7 8463	1.5607	8.8271	1.7558	7
1¼	6.7902	1.7009	7.7602	1.9438	8.7303	2.1868	6¾
1½	6.6986	2.0320	7.6555	2.3223	8.6125	2.6126	6½
1¾	6.5908	2.3582	7.5324	2.6951	8.4739	3.0320	6¼
2	6 4672	2.6788	7.3910	3.0615	8.3149	3.4442	6
2¼	6.3279	2.9929	7.2319	3.4204	8.1359	3.8480	5¾
2½	6.1734	3.2998	7.0554	3.7712	7.9373	4.2426	5½
2¾	6.0041	3.5987	6.8618	4.1128	7.7196	4.6269	5¼
3	5.8203	3.8890	6.6518	4.4446	7.4832	5.0001	5
3¼	5 6225	4.1699	6.4257	4.7656	7.2289	5.3613	4¾
3½	5.4111	4.4408	6.1841	5.0751	6.9571	5.7095	4½
3¾	5.1867	4.7009	5 9276	5.3725	6.6686	6.0440	4¼
4	4.9497	4.9497	5.6569	5.6569	6.3640	6.3640	4
Pts.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Pts.

TAB. C.—Log. Sines, Tangents, Secants, &c., to every Point and Quarter Point of the Compass.

Points.	Sine.	Cosine.	Tangent.	Cotangent.	Secant.	Cosecant.	Points.
0	0.000000	10.000000	0.000000	Infinite.	10.000000	Infinite.	8
0¼	8.690796	9.999477	8.691319	11.308681	10.000523	11.309204	7¾
0½	8.991302	9.997904	8.993398	11.006602	10.002096	11.008698	7½
0¾	9.166520	9.995274	9.171247	10.828753	10.004726	10.833480	7¼
1	9.290236	9.991574	9.298662	10.701338	10.008426	10.709764	7
1¼	9.385571	9.986786	9.398785	10.601215	10.013214	10.614429	6¾
1½	9.462824	9.980885	9.481939	10.518061	10.019115	10.537176	6½
1¾	9.527488	9.973841	9.553647	10.446353	10.026159	10.472512	6¼
2	9.582840	9.965615	9.617224	10.382776	10.034385	10.417160	6
2¼	9.630992	9.956163	9.674829	10.325171	10.043837	10.369008	5¾
2½	9.673387	9.945430	9.727957	10.272043	10.054570	10.326613	5½
2¾	9.711050	9.933350	9.777700	10.222300	10.066650	10.288950	5¼
3	9.744739	9.919846	9.824893	10.175107	10.080154	10.255261	5
3¼	9.775027	8.904828	9.870199	10.129801	10.095172	10.224973	4¾
3½	9.802359	9.888185	9.914173	10.085827	10.111815	10.197641	4½
3¾	9.827084	9.869790	9.957295	10.042705	10.130210	10.172916	4¼
4	9.849485	9.849485	10.000000	10.000000	10.150515	10.150515	4
Points.	Cosine.	Sine.	Cotangent.	Tangent.	Cosecant.	Secant.	Points.

TAB. D.—Lengths of a Degree of Longitude on the parallels of Latitude, for each degreee of latitude from the Equator to the Pole. Formula, $L = 60 \cos lat.$

Lat.	Miles.	Lat.	Miles.	Lat.	Miles.	Lat.	Miles.	Lat.	Miles.
1°	59.99	19°	56.73	37°	47.92	55°	34.41	73°	17.54
2	59.96	20	56.38	38	47.28	56	33.45	74	16.54
3	59 92	21	56.01	39	46.63	57	32.68	75	15.53
4	59.85	22	55.63	40	45.96	58	31.79	76	14.52
5	59.77	23	55.23	41	45.28	59	30.90	77	13.50
6	59.67	24	54.81	42	44.59	60	30.00	78	12.48
7	59.55	25	54.38	43	43.88	61	29.09	79	11.45
8	59.42	26	53.93	44	43.16	62	28.17	80	10.42
9	59.26	27	53.46	45	42.43	63	27.74	81	9.38
10	59.09	28	52.97	46	41.68	64	26.30	82	8.35
11	58.89	29	52.48	47	40.92	65	25.36	83	7.31
12	58.69	30	51.96	48	40.15	66	24.40	84	6.27
13	58.46	31	51.43	49	39.36	67	23.44	85	5.23
14	58.22	32	50.88	50	38.57	68	22.48	86	4.18
15	57.95	33	50.32	51	37.76	69	21.50	87	3.14
16	57.67	34	49.74	52	36.94	70	20.52	88	2.09
17	57.38	35	49.15	53	36.11	71	19.53	89	1.05
18	57.06	36	48.54	54	35.27	72	18.54	90	0.00

TAB. E.—The SUN'S PARALLAX in ALTITUDE,
For the 1st Day of every Month in the Year.

Alt.	Zen. dist.	Jan.	Feb. Dec.	Mar. Nov.	Apr. Oct.	May Sep.	June Aug.	July.
80°	10°	1″.55	1″.54	1″.53	1.″52	1″.51	1″.50	1″.50
79	11	1 .71	1 .70	1 .69	1 .68	1 .66	1 .65	1 .65
78	12	1 .86	1 .85	1 .84	1 .83	1 .81	1 .80	1 .80
77	13	2 .02	2 .01	1 .99	1 .98	1 .96	1 .95	1 .95
76	14	2 .17	2 .16	2 .14	2 .13	2 .11	2 .10	2 .09
75	15	2 .32	2 .31	2 .29	2 .28	2 .26	2 .25	2 .24
74	16	2 .47	2 .46	2 .44	2 .43	2 .41	2 .39	2 .38
73	17	2 .62	2 .61	2 .59	2 .58	2 .56	2 .54	2 .53
72	18	2 .77	2 .76	2 .74	2 .72	2 .70	2 .68	2 .67
71	19	2 .92	2 .91	2 .89	2 .87	2 .85	2 .83	2 .82
70	20	3 .06	3 .05	3 .03	3 .01	2 .99	2 .97	2 .96
69	21	3 .21	3 .20	3 .18	3 .16	3 .13	3 .11	3 .10
68	22	3 .35	3 .34	3 .32	3 .30	3 .27	3 .25	3 .24
67	23	3 .50	3 .49	3 .47	3 .44	3 .41	3 .39	3 .38
66	24	3 .64	3 .63	3 .61	3 .58	3 .55	3 .53	3 .52
64	26	3 .92	3 .91	3 .89	3 .86	3 .83	3 .80	3 .79
62	28	4 .20	4 .19	4 .16	4 .13	4 .10	4 .07	4 .06
60	30	4 .47	4 .46	4 .43	4 .40	4 .36	4 .33	4 .32
58	32	4 .74	4 .73	4 .70	4 .66	4 .63	4 .59	4 .58
56	34	5 .00	4 .99	4 .96	4 .92	4 .88	4 .84	4 .83
54	36	5 .26	5 .25	5 .21	5 .17	5 .13	5 .09	5 .08
52	38	5 .51	5 .50	5 .46	5 .42	5 .37	5 .33	5 .32
50	40	5 .75	5 .74	5 .70	5 .66	5 .61	5 .57	5 .56
48	42	5 .99	5 .97	5 .93	5 .88	5 .84	5 .80	5 .79
Alt.	Zen. dist.	Jan.	Feb. Dec.	Mar. Nov.	Apr. Oct.	May Sep.	June Aug.	July.

TAB. E.—The SUN'S PARALLAX in ALTITUDE,
For the 1st Day of every Month in the Year.

Alt.	Zen. dist.	Jan.	Feb. Dec.	Mar. Nov.	Apr. Oct.	May Sep.	June Aug.	July.
46°	44°	6″.22	6″.20	6″.16	6″.11	6″.06	6″.02	6″.01
44	46	6 .44	6 .42	6 .38	6 .33	6 .28	6 .23	6 .22
42	48	6 .65	6 .64	6 .59	6 .54	6 .49	6 .44	6 .43
40	50	6 .85	6 .84	6 .79	6 .74	6 .69	6 .64	6 .62
38	52	7 .05	7 .04	6 .99	6 .93	6 .88	6 .83	6 .82
36	54	7 .24	7 .22	7 .17	7 .11	7 .06	7 .01	7 .00
34	56	7 .42	7 .40	7 .[illegible]5	7 .29	7 .23	7 .18	7 .17
32	58	7 .59	7 .57	7 .[illegible]2	7 .46	7 .40	7 .35	7 .33
30	60	7 .75	7 .73	7 .68	7 .62	7 .56	7 .51	7 .49
28	62	7 .90	7 .88	7 .83	7 .77	7 .71	7 .65	7 .63
26	64	8 .04	8 .03	7 .97	7 .91	7 .85	7 .79	7 .77
24	66	8 .17	8 .16	8 .10	8 .04	7 .97	7 .92	7 .90
22	68	8 .30	8 .28	8 .22	8 .16	8 .09	8 .04	8 .02
20	70	8 .41	8 .39	8 .33	8 .27	8 .20	8 .15	8 .13
18	72	8 .51	8 .49	8 .44	8 .37	8 .30	8 .25	8 .23
16	74	8 .60	8 .58	8 .53	8 .46	8 .39	8 .33	8 .31
14	76	8 .68	8 .66	8 .61	8 .54	8 .47	8 .41	8 .39
12	78	8 .75	8 .73	8 .67	8 .61	8 .54	8 .48	8 .46
10	80	8 .81	8 .79	8 .73	8 .67	8 .60	8 .54	8 .52
8	82	8 .86	8 .84	8 .78	8 .71	8 .64	8 .58	8 .56
6	84	8 .90	8 .88	8 .82	8 .75	8 .68	8 .62	8 .60
4	86	8 .92	8 .90	8 .84	8 .77	8 .70	8 .64	8 .62
2	88	8 .94	8 .92	8 .86	8 .79	8 .72	8 .66	8 .64
0	90	8 .95	8 .93	8 .87	8 .80	8 .73	8 .67	8 .65
Alt.	Zen. dist.	Jan.	Feb. Dec.	Mar. Nov.	Apr. Oct.	May Sep.	June Aug.	July.

TAB. F.—The Moon's Parallax in Altitude, Corrected for Refraction, and Horizontal Parallax.

Alt.	53′	54′	55′	56′	57′	58′	59′	60′	61′
	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″	′ ″
3°	38 21	39 21	40 20	41 20	42 20	43 20	44 20	45 20	46 20
4	41 2	42 2	43 2	44 2	45 1	46 1	47 1	48 1	49 1
5	42 55	43 55	44 54	45 54	46 54	47 54	48 54	49 53	50 53
6	44 15	45 16	46 15	47 15	48 15	49 14	50 14	51 14	52 13
7	45 16	46 15	47 14	48 14	49 13	50 13	51 13	52 12	53 12
8	46 0	46 59	47 58	48 58	49 57	50 56	51 56	52 55	53 55
9	46 33	47 31	48 31	49 30	50 29	51 29	52 28	53 27	54 26
10	46 57	47 56	48 55	49 54	50 53	51 52	52 51	53 50	54 49
11	47 14	48 13	49 12	50 11	51 10	52 9	53 8	54 7	55 6
12	47 27	48 26	49 25	50 23	51 22	52 21	53 19	54 18	55 17
13	47 35	48 34	49 32	50 31	51 29	52 28	53 26	54 25	55 23
14	47 40	48 38	49 36	50 35	51 33	52 31	53 29	54 27	55 26
15	47 41	48 39	49 37	50 35	51 33	52 31	53 29	54 27	55 25
16	47 30	48 28	49 25	50 23	51 21	52 18	53 16	54 14	55 11
17	47 27	48 24	49 21	50 18	51 16	52 13	53 11	54 8	55 5
18	47 20	48 17	49 14	50 12	51 9	52 6	53 3	54 0	54 57
19	47 13	48 9	49 6	50 3	50 59	51 56	52 53	53 50	54 46
20	47 3	47 59	48 55	49 52	50 48	51 45	52 41	53 37	54 34
21	46 52	47 47	48 43	49 39	50 35	51 31	52 27	53 23	54 19
22	46 39	47 34	48 29	49 25	50 21	51 16	52 12	53 8	54 3
23	46 23	47 19	48 14	49 9	50 4	51 0	51 55	52 50	53 45
24	46 7	47 2	47 57	48 52	49 47	50 42	51 37	52 31	53 26
25	45 50	46 45	47 39	48 33	49 28	50 22	51 17	52 11	53 5
26	45 32	46 26	47 20	48 13	49 7	50 1	50 55	51 49	52 43
27	45 12	46 5	46 59	47 52	48 46	49 39	50 33	51 26	52 20
28	44 51	45 44	46 37	47 30	48 23	49 16	50 9	51 2	51 55
29	44 29	45 21	46 14	47 6	47 59	48 51	49 44	50 36	51 29
30	44 6	44 57	45 49	46 41	47 33	48 25	49 17	50 9	51 1
31	43 41	44 33	45 24	46 15	47 7	47 58	48 50	49 41	50 33
32	43 15	44 7	44 58	45 48	46 39	47 30	48 21	49 12	50 3
33	42 50	43 40	44 30	45 20	46 11	47 1	47 51	48 42	49 32
34	42 22	43 12	44 2	44 51	45 41	46 31	47 20	48 10	49 0
35	41 54	42 43	43 32	44 21	45 10	45 59	46 49	47 38	48 27
36	41 25	42 13	43 1	43 50	44 39	45 27	46 16	47 4	47 53
37	40 54	41 42	42 30	43 18	44 6	44 54	45 52	46 30	47 18
38	40 23	41 10	41 58	42 45	43 32	44 19	45 7	45 54	46 41
39	39 52	40 38	41 24	42 11	42 58	43 44	44 31	45 17	46 4
40	39 18	40 4	40 50	41 36	42 22	43 8	43 54	44 40	45 26
41	38 45	39 30	40 15	41 0	41 46	42 31	43 16	44 1	44 47

TAB. F.—The Moon's Parallax in Altitude, Corrected for Refraction, and Horizontal Parallax.

Alt.	53′	54′	55′	56′	57′	58′	59′	60′	61′
42°	38′ 10″	38′ 55″	39′ 39″	40′ 24″	41′ 8″	41′ 53″	42′ 38″	43′ 22″	44′ 7″
43	37 35	38 19	39 2	39 46	40 30	41 14	41 58	42 42	43 26
44	36 58	37 42	38 25	39 8	39 51	40 34	41 17	42 1	42 44
45	36 22	37 4	37 46	38 29	39 11	39 54	40 36	41 19	42 1
46	35 44	36 27	37 7	37 49	38 31	39 12	39 54	40 36	41 17
47	35 6	35 47	36 28	37 9	37 49	38 30	39 11	39 52	40 33
48	34 27	35 7	35 47	36 27	37 7	37 47	38 27	39 8	39 48
49	33 47	34 26	35 5	35 45	36 24	37 4	37 43	38 22	39 2
50	33 6	33 45	34 23	35 2	35 41	36 19	36 58	37 36	38 15
51	32 25	33 3	33 41	34 18	34 56	35 34	36 12	36 49	37 27
52	31 43	32 20	32 57	33 34	34 11	34 48	35 25	36 2	36 39
53	31 1	31 37	32 13	32 49	33 25	34 1	34 38	35 14	35 50
54	30 18	30 53	31 28	32 3	32 39	33 14	33 49	34 25	35 0
55	29 34	30 9	30 43	31 17	31 52	32 26	33 1	33 35	34 9
56	28 50	29 23	29 57	30 30	31 4	31 38	32 11	32 45	33 18
57	28 5	28 38	29 10	29 43	30 16	30 48	31 21	31 54	32 26
58	27 20	27 51	28 23	28 55	29 27	29 59	30 30	31 2	31 34
59	26 34	27 5	27 35	28 6	28 37	29 8	29 39	30 10	30 41
60	25 47	26 17	26 47	27 17	27 47	28 17	28 47	29 17	29 47
61	25 0	25 29	25 58	26 27	26 56	27 26	27 55	28 24	28 53
62	24 13	24 41	25 9	25 37	26 5	26 33	27 2	27 30	27 58
63	23 25	23 52	24 19	24 46	25 14	25 41	26 8	26 35	27 3
64	22 36	23 3	23 29	23 55	24 21	24 48	25 14	25 40	26 7
65	21 47	22 13	22 38	23 3	23 29	23 54	24 20	24 45	25 10
66	20 59	21 22	21 47	22 11	22 36	23 0	23 24	23 49	24 13
67	20 9	20 32	20 55	21 19	21 42	22 6	22 29	22 52	23 16
68	19 18	19 41	20 3	20 26	20 48	21 11	21 33	21 56	22 18
69	18 28	18 49	19 11	19 32	19 54	20 15	20 37	20 58	21 20
70	17 37	17 57	18 18	18 38	18 59	19 20	19 40	20 1	20 21
71	16 46	17 5	17 25	17 44	18 4	18 23	18 43	19 2	19 22
72	15 54	16 13	16 31	16 50	17 8	17 27	17 45	18 4	18 22
73	15 3	15 20	15 37	15 55	16 12	16 30	16 48	17 5	17 23
74	14 11	14 27	14 43	15 0	15 16	15 33	15 49	16 6	16 23
75	13 18	13 33	13 49	14 4	14 20	14 35	14 51	15 6	15 22
76	12 25	12 40	12 54	13 9	13 23	13 38	13 52	14 7	14 21
77	11 32	11 46	11 59	12 13	12 26	12 40	12 53	13 7	13 20
78	10 39	10 52	11 4	11 16	11 29	11 41	11 54	12 6	12 19
79	9 46	9 57	10 9	10 20	10 31	10 43	10 54	11 6	11 17
80	8 52	9 3	9 13	9 23	9 34	9 44	9 55	10 5	10 15

TAB. G.—Mean Refraction of the Heavenly Bodies.
Apparent Altitudes from 0° to 8° 15′.

Alt.	Refrac.	Alt.	Refrac.	Alt.	Refrac.	Alt.	Refr.
° ′	′ ″	° ′	′ ″	° ′	′ ″	° ′	′ ″
0 0	33 0	2 5	18 11	4 10	11 29	6 15	8 9
0 5	32 10	2 10	17 48	4 15	11 18	6 20	8 3
0 10	31 22	2 15	17 26	4 20	11 8	6 25	7 57
0 15	30 35	2 20	17 4	4 25	10 58	6 30	7 51
0 20	29 50	2 25	16 44	4 30	10 48	6 35	7 45
0 25	29 6	2 30	16 24	4 35	10 39	6 40	7 40
0 30	28 33	2 35	16 4	4 40	10 29	6 45	7 35
0 35	27 41	2 40	15 45	4 45	10 20	6 50	7 30
0 40	27 0	2 45	15 27	4 50	10 11	6 55	7 25
0 45	26 20	2 50	15 9	4 55	10 2	7 0	7 20
0 50	25 42	2 55	14 52	5 0	9 54	7 5	7 15
0 55	25 5	3 0	14 36	5 5	9 46	7 10	7 11
1 0	24 29	3 5	14 20	5 10	9 38	7 15	7 6
1 5	23 54	3 10	14 4	5 15	9 30	7 20	7 2
1 10	23 20	3 15	13 49	5 20	9 23	7 25	6 57
1 15	22 47	3 20	13 34	5 25	9 15	7 30	6 53
1 20	22 15	3 25	13 20	5 30	9 8	7 35	6 49
1 25	21 44	3 30	13 6	5 35	9 1	7 40	6 45
1 30	21 15	3 35	12 53	5 40	8 54	7 45	6 41
1 35	20 46	3 40	12 40	5 45	8 47	7 50	6 37
1 40	20 18	3 45	12 27	5 50	8 41	7 55	6 33
1 45	19 51	3 50	12 15	5 55	8 34	8 0	6 29
1 50	19 25	3 55	12 3	6 0	8 28	8 5	6 25
1 55	19 0	4 0	11 51	6 5	8 21	8 10	6 22
2 0	18 35	4 5	11 40	6 10	8 15	8 15	6 18

TAB. G.—Mean Refraction of the Heavenly Bodies. Apparent Altitudes from 8° 20′ to 23° 10′.

Alt.	Ref.	Alt.	Ref.	Alt.	Ref.	Alt.	Ref.
° ′	′ ″	° ′	′ ″	° ′	′ ″	° ′	′ ″
8 20	6 15	10 50	4 51	15 0	3 30	19 10	2 43
8 25	6 11	11 0	4 47	15 10	3 28	19 20	2 41
8 30	6 8	11 10	4 43	15 20	3 26	19 30	2 40
8 35	6 5	11 20	4 39	15 30	3 24	19 40	2 38
8 40	6 1	11 30	4 34	15 40	3 21	19 50	2 37
8 45	5 58	11 40	4 31	15 50	3 19	20 0	2 36
8 50	5 55	11 50	4 27	16 0	3 17	20 10	2 35
8 55	5 52	12 0	4 23	16 10	3 15	20 20	2 32
9 0	5 48	12 10	4 20	16 20	3 12	20 30	2 31
9 5	5 45	12 20	4 16	16 30	3 10	20 40	2 29
9 10	5 42	12 30	4 13	16 40	3 8	20 50	2 28
9 15	5 39	12 40	4 9	16 50	3 6	21 0	2 27
9 20	5 36	12 50	4 6	17 0	3 4	21 10	2 26
9 25	5 34	13 0	4 3	17 10	3 3	21 20	2 25
9 30	5 31	13 10	4 0	17 20	3 1	21 30	2 24
9 35	5 28	13 20	3 57	17 30	2 59	21 40	2 23
9 40	5 25	13 30	3 54	17 40	2 57	21 50	2 21
9 45	5 23	13 40	3 51	17 50	2 55	22 0	2 20
9 50	5 20	13 50	3 48	18 0	2 54	22 10	2 19
9 55	5 18	14 0	3 45	18 10	2 52	22 20	2 18
10 0	5 15	14 10	3 43	18 20	2 51	22 30	2 17
10 10	5 10	14 20	3 40	18 30	2 49	22 40	2 16
10 20	5 5	14 30	3 38	18 40	2 47	22 50	2 15
10 30	5 0	14 40	3 35	18 50	2 46	23 0	2 14
10 40	4 56	14 50	3 33	19 0	2 44	23 10	2 13

TAB. G.—MEAN REFRACTION of the Heavenly Bodies. Apparent Altitudes from 23° 20′ to 90°.

Alt.	Ref.	Alt.	Ref.	Alt.	Ref.	Alt.	Refr.
° ′	′ ″	° ′	′ ″	° ′	′ ″	°	′ ″
23 20	2 12	27 45	1 48	39 0	1 10	63	0 29
23 30	2 11	28 0	1 47	39 30	1 9	64	0 28
23 40	2 10	28 15	1 46	40 0	1 8	65	0 26
23 50	2 9	28 30	1 45	41 0	1 5	66	0 25
24 0	2 8	28 45	1 44	42 0	1 3	67	0 24
24 10	2 7	29 0	1 42	43 0	1 1	68	0 23
24 20	2 6	29 30	1 40	44 0	0 59	69	0 22
24 30	2 5	30 0	1 38	45 0	0 57	70	0 21
24 40	2 4	30 30	1 37	46 0	0 55	71	0 19
24 50	2 3	31 0	1 35	47 0	0 53	72	0 18
25 0	2 2	31 30	1 33	48 0	0 51	73	0 17
25 10	2 1	32 0	1 31	49 0	0 49	74	0 16
25 20	2 0	32 30	1 30	50 0	0 48	75	0 15
25 30	1 59	33 0	1 28	51 0	0 46	76	0 14
25 40	1 58	33 30	1 26	52 0	0 44	77	0 13
25 50	1 57	34 0	1 24	53 0	0 43	78	0 12
26 0	1 56	34 30	1 23	54 0	0 41	79	0 11
26 10	1 55	35 0	1 21	55 0	0 40	80	0 10
26 20	1 55	35 30	1 20	56 0	0 38	81	0 9
26 30	1 54	36 0	1 18	57 0	0 37	82	0 8
26 40	1 53	36 30	1 17	58 0	0 35	83	0 7
26 50	1 52	37 0	1 16	59 0	0 34	84	0 6
27 0	1 51	37 30	1 14	60 0	0 33	86	0 4
27 15	1 50	38 0	1 13	61 0	0 32	88	0 2
27 30	1 49	38 30	1 11	62 0	0 30	90	0 0

TAB. H.—Dip of the Horizon, corrected for Refraction.

Height of Eye.	Dip.	Height of Eye.	Dip.	Height of Eye.	Dip.
ft.	′ ″	ft.	′ ″	ft.	′ ″
1	0 59	28	5 13	70	8 14
2	1 24	29	5 18	71	8 18
3	1 42	30	5 24	74	8 28
4	1 58	31	5 29	77	8 38
5	2 12	32	5 34	80	8 48
6	2 25	33	5 39	83	8 58
7	2 36	34	5 44	86	9 8
8	2 47	35	5 49	89	9 17
9	2 57	36	5 54	90	9 22
10	3 7	37	5 59	92	9 26
11	3 16	38	6 4	95	9 36
12	3 25	39	6 9	98	9 45
13	3 33	40	6 14	100	9 52
14	3 41	41	6 18	101	9 54
15	3 49	42	6 23	104	10 2
16	3 56	43	6 27	107	10 11
17	4 4	44	6 32	110	10 19
18	4 11	45	6 37	113	10 28
19	4 17	47	6 45	116	10 36
20	4 24	48	6 50	119	10 44
21	4 31	50	6 58	122	10 52
22	4 37	53	7 10	125	11 0
23	4 43	56	7 22	128	11 8
24	4 49	59	7 34	131	11 16
25	4 55	62	7 45	134	11 24
26	5 1	65	7 56	137	11 31
27	5 7	68	8 7	140	11 39

TAB. I.—Dip of the Horizon, at various distances from it.

Dist. in miles.	Height of the eye, ft. 5	ft. 10	ft. 15	ft. 20	ft. 25	ft. 30
0¼	11′	22′	34′	45′	56′	68′
0½	6	11	17	22	28	34
0¾	4	8	12	16	19	23
1	4	6	9	12	15	17
1¼	3	5	7	9	12	14
1½	3	4	6	8	10	12
2	2	3	5	6	8	10
2½	2	3	5	6	7	8
3	2	3	4	5	6	7
3½	2	3	4	5	6	6
4	2	3	4	4	5	6
5	2	3	4	4	5	5
6	2	3	4	4	5	5

TAB. K.—Augmentation of Moon's semidiameter.

Alt.	14′½	15′½	16′½	Alt.	14′½	15′½	16′½
	″	″	″		″	″	″
6°	1	2	2	39°	9	10	11
9	2	2	3	42	9	10	12
12	3	3	4	45	10	11	12
15	4	4	5	48	10	12	13
18	4	5	5	51	11	12	14
21	5	6	6	54	11	13	14
24	6	6	7	57	11	13	15
27	6	7	8	60	12	13	15
30	7	8	9	66	12	14	16
33	7	8	10	75	13	15	17
36	8	9	10	90	14	16	18

TAB. L.—Difference between True and Apparent Levels *in Feet.*

Distance in Miles.	Diff. for Curvature.	Corrected for Refrac.	Distance in Chains.	Diff. for Curvature.	Corrected for Refrac.
0¼	0.042	0.036	1	.000104	.000089
0½	0.167	0.142	2	.000417	.000358
0¾	0.375	0.322	3	.000938	.000804
1	0.667	0.571	4	.001668	.001430
2	2.667	2.285	5	.002605	.002233
3	6.000	5.142	6	.003752	.003216
4	10.675	9.150	7	.005107	.004378
5	16.675	14.292	8	.006670	.005717
6	24.008	20.578	9	.008442	.007236
7	32.683	28.014	10	.010422	.008933
8	42.692	36.593	11	.012610	.010809
9	54.025	46.307	12	.015007	.012863
10	66.700	57.171	13	.017613	.015097
11	80.708	69.178	14	.020427	.017509
12	96.050	82.329	15	.023450	.020100
13	112.717	96.615	16	.026680	.022869
14	130.733	112.057	17	.030120	.025817
15	150.075	128.636	18	.033767	.028943
16	170.750	146.357	19	.037623	.032248
17	192.767	165.229	20	.041687	.035732
18	216.108	185.235	21	.045960	.039394
19	240.783	206.385	22	.050442	.043236
20	266.800	228.686	23	.055132	.047259
			24	.060031	.051455
Ft.			25	.065137	.055832
500	0.00598	0.00513			
1000	0.02392	0.02050	26	.070452	.060388
2000	0.09570	0.08206	27	.075975	.065121
3000	0.21533	0.18457	28	.081708	.070036
4000	0.38281	0.32812	29	.087648	.075127
5000	0.59814	0.51269	30	.093798	.080399

TAB. M.—Distance of Horizon corrected for Refraction.

Height in Feet.	Distance in Nautical Miles.
10	3.6
20	5.1
30	6.3
40	7.3
50	8.1
60	8.9
70	9.6
80	10.3
90	10.9
100	11.5
200	16.2
300	19.9
400	23.0
500	25.7
600	28.1
700	30.4
800	32.5
900	34.5
1000	36.
2000	51.
3000	63.
4000	72.
5000	81.
6000	89.
7000	96.
8000	103.
9000	109.
10000	115.
20000	163
30000	199

Glasgow:—E. Khull, Printer to the University.